Statistics
Bivariate Normal Distribution
(Part-2)
Conditional Distributions and Results

1. Introduction

Welcome to the series of E-learning modules on Bivariate Normal Distribution- conditional distributions and results.

By the end of this session, you will be able to:

- Explain the conditional distribution
- Explain the coefficient of correlation
- Explain the recurrence relation of central moments

In the previous module, we have discussed about the bivariate normal distribution, its probability density function and marginal distributions. Hence, we know that the probability density function of bivariate normal distribution is given by, f XY of x, y is equal to 1 divided by 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square, into x minus mu 1 the whole square divided by sigma 1 square, minus 2 into rho into x minus mu 1 into y minus mu 2 divided by sigma 1 into sigma 2 plus y minus mu 2 whole square divided by sigma 2 square.

Its marginal probability density functions are given by, fX of x is equal to 1 divided by sigma 1 into square root of 2 into phi, into e power minus half into x minus mu 1 divided by sigma 1 the whole square and fY of y is equal to 1 divided by sigma 2 into square root of 2 into phi, into e power minus half into x minus mu 2 divided by sigma 2 the whole square.

2. Conditional Distribution

Now, let us obtain conditional distribution of X. Conditional distribution of X for fixed Y is given by,

f X given Y of x given y is equal to f X Y of x, y divided by fY of y is equal to 1 divided by 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square, into x minus mu 1 the whole square divided by sigma 1 square, minus 2 into rho into x minus mu 1 into y minus mu 2 divided by sigma 1 into sigma 2 plus y minus mu 2 whole square divided by sigma 2 square, divided by, 1 divided by sigma 2 into square root of 2 into phi, into e power minus half into x minus mu 2 divided by sigma 2 the whole square. Is equal to 1 divided by square root of 2 into phi into sigma 1 into square root of 1 minus rho square into, e power minus 1 divided by 2 into 1 minus rho square, into x minus mu 1 the whole square divided by sigma 1 square, minus 2 into rho into x minus mu 1 into y minus mu 2 divided by sigma 1 into sigma 2, plus y minus mu 2 whole square divided by sigma 2 square, into 1 minus, 1 minus rho square.

Is equal to 1 divided by square root of 2 into phi into sigma 1 into square root of 1 minus rho square into, e power minus 1 divided by 2 into 1 minus rho square into sigma 1 square, into x minus mu 1, minus rho into sigma 1 divided by sigma 2 into y minus mu 2 the whole square. Is equal to 1 divided by square root of 2 into phi into sigma 1 into square root of 1 minus rho square into, e power minus 1 divided by 2 into 1 minus rho square into sigma 1 square, into x minus, mu 1 plus rho into sigma 1 divided by sigma 2 into y minus mu 2 the whole square. which is the probability function of a univariate normal distribution with mean and variance, which is given by, Expectation of X given Y is equal to mu 1 plus rho into sigma 1 divided by sigma 2 into y minus mu 2 and variance of X given Y is equal to 1 minus rho square into sigma 1 square. Hence, conditional distribution of X for fixed Y is also normal given by, X given Y is equal to y follows Normal distribution wit parameters (mu 1 plus rho into sigma 1 divided by sigma 2 into y minus mu 2 and 1 minus rho square into sigma 1 square).

Similarly, the conditional distribution of Y for fixed X is given by, fY given X of y given x is equal to fX Y of x, y divided by f x of x Is equal to 1 divided by square root of 2 into phi into sigma 2 into square root of 1 minus rho square, into e power minus 1 divided by 2 into 1 minus rho square into sigma 2 square, into y minus, mu2 plus rho into sigma 2 divided by sigma 1 into x minus mu 1 the whole square
Thus, conditional distribution of Y for fixed X is also normal and given by, Y given X is equal to x follows Normal distribution with parameters (mu2 plus rho into sigma 2 divided by sigma 1 into x minus mu 1 and 1 minus rho square into sigma 2 square.

It is apparent from the above results that the array mean are collinear. That is the regression equations are linear (involving linear functions of the independent variables) and the array variance are constant (that is free from independent variable). We express this by saying that the regression equations of Y on X and X on Y are linear and homoscedastic.

For rho is equal to zero, the conditional variance V of (Y given X) is equal to the marginal variance sigma 2 square and the conditional mean of Expectation of Y given X is equal to the marginal mean mu 2 and the two variables become independent, which is also apparent from joint distribution function. In between the two extremes when rho is equal to plus or minus 1, the coefficient rho provides a measure of degree of association or interdependence between the two variables.

3. Coefficient of Correlation

Now, consider the following results. Show that if X and Y are standard normal variates with correlation coefficient rho between them, then the correlation coefficient between X square and Y square is given by rho square.

We prove this result as follows.
Given X and Y are two standard normal variates. Hence, we have, Expectation of X is equal to Expectation of Y is equal to zero. Variance of X is equal to Expectation of X square is equal to 1, which is equal to Variance of Y is equal to Expectation of Y square. Therefore, M X, Y of t1, t2 is equal to e power half into t1 square plus 2 into rho into t1 into t2 plus t2 square.

Now, consider the coefficient of correlation, Rho of x square, Y square Is equal to expectation of X 1 square into X 2 square minus expectation of X1 square into Expectation of X2 square whole divided by square root of Expectation of X1 power 4 minus expectation of X1 square the whole square into expectation of X2 power 4 minus Expectation of X2 square the whole square, where Expectation of X1 square into X2 square is equal to coefficient of t1 square divided by 2 factorial into t2 square divided by 2 factorial in M of t1, t2 Is equal to 2 into rho square plus 1 Expectation of X1 power 4 is equal to coefficient of t1 power 4 divided by 4 factorial in M of t1, t2 is equal to 3. Expectation of X2 power 4 is equal to coefficient of t2 power 4 divided by 4 factorial in M of t1, t2 is equal to 3. By substituting the different values, in rho X square, Y square, we get, Rho X square Y square is equal to 2 into rho square plus 1 minus 1 into 1 divided by square root of 3 minus 1 into 3 minus 1 Is equal to rho square. Therefore, the correlation coefficient between X square and Y square is given by rho square.

Consider the 2^{nd} result. If X and Y are standard normal variates with coefficient of correlation rho, then show that, Q is equal to (X square plus 2 into rho into X into Y plus Y square) divided by (1 minus rho square) is distributed like a chi-square, that is as that of the sum of the squares of standard normal variates.

To prove this result, let us consider the moment generating function of Q M Q of t is equal to Expectation of e power t into Q Is equal to double integral from minus infinity to infinity, e power t into Q f of x, y dx dy.

Is equal to 1 divided by 2 into phi into square root of 1 minus rho square into double integral from minus infinity to infinity e power t into Q into e power minus 1 divided by 2 into 1 minus rho square into x square minus 2 into rho into x into y plus y square, dx, dy Is equal to 1 divided by 2 into phi into square root of 1 minus rho square into double integral from minus infinity to infinity e power t into Q minus Q divided by 2 dx, dy Is equal to 1 divided by 2 into phi into square root of 1 minus rho square into double integral from minus infinity to infinity e power minus Q divided by 2 into 1 minus 2 into t dx, dy. Put x into square root of 1 minus 2 into t is equal to u and y into square root of 1 minus 2 into t is equal to v
Implies dx is equal to du divided by square root of 1 minus 2 into t and dy is equal to dv divided by square root of 1 minus 2 into t. Also, Q is equal to 1 divided by 1 minus rho square into x square minus 2 into rho into x into y plus y square Is equal to 1 divided by 1 minus rho square into u square minus 2 into rho into u into v plus v square divided by 1 minus 2 into t.

Therefore, M Q of t is equal to 1 divided by 2 into phi into 1 minus 2 into t into square root of 1 minus rho square, double integral from minus infinity to infinity e power minus 1 divided by 2 into 1 minus rho square into u square minus 2 into rho into u into v plus v square du, dv. Is equal to 1 divided by 1 minus 2 into t into 1, is equal to 1 minus 2 into t whole power minus 1, Which is the moment generating function of Chi-square variate with n (equal to 2) degrees of freedom. (We study about chi-square distribution in the later modules.)

4. Recurrence Relation of Central Moments

Show that for a bivariate normal distribution with probability function f of x, y is equal to 1 divided by 2 into phi into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square, into x square minus 2 into rho into x into y plus y square, the moments obey the recurrence relation mu r, s is equal to r plus s minus 1 into rho into mu r minus 1, s minus 1 plus r minus 1 into s minus 1 into 1 minus rho square into mu r minus 2, s minus 2.

Let us prove this result using moment generating function.
We know that moment generating function of above distribution is given by,
M is equal to M of t1, t2 is equal to e power half into t1 square plus 2 into rho into t1 into t2 plus t2 square

Now, finding partial differentiation, we get, Dho M by dho t1 is equal to M of t1 plus rho into t2
Dho M by dho t2 is equal to M of t2 plus rho into t1 Dho square M by dho t1, dho t2 is equal to dho by dho t1 of dho M by dho t2 Is equal to dho by dho t1, M of t2 plus rho t1 Is equal to M into rho plus t2 plus rho t1 into t1 plus rho t2 into M

Therefore, dho squre M by dho t1, dho t2 minus rho into t1 into dho M dho t1 minus rho into t2 into dho M by dho t2 Is equal to M into rho plus t2 plus rho into t1, into t1 into rho into t2, into M, minus rho into t1, into t1 plus rho into t2 into M minus rho into t2, into t2 plus rho into t1 into M On simplification, we get, M into t1 into t2 plus rho minus rho square into t1 into t2 Is equal to M into rho plus 1 minus rho square into M into t1, t2.

Therefore, dho square M by dho t1,dho t2 is equal to rho into t1 into dho M by dho t1 plus rho into t2 into dho M by dho t2 plus M into rho plus 1 minus rho square into M into t1 into t2. Name it as star. But M is equal to e power half into t1 square plus 2 into rho into t1 into t2 plus t2 square Is equal to summation from r is equal to zero to infinity, summation from s is equal to zero to infinity mu r, s into t1 power r into t2 power s divided by r factorial into s factorial

Therefore, star equation gives
Summation over r is equal to 1 to infinity, summation over s is equal to 1 to infinity mu r, s into t1 power r minus 1 into t2 power s minus 1 divided by r minus 1 factorial into s minus 1 factorial Is equal to rho into summation over r is equal to 1 to infinity, s is equal to zero to infinity r into mu r,s t1 power r into t2 power s divided by r factorial into s factorial plus rho into summation over r is equal to zero to infinity, summation over s is equal to 1 to infinity s into mu r,s t1 power r into t2 power s divided by r factorial into s factorial plus rho into summation over r is equal to zero to zero to infinity, s is equal to zero infinity mu r,s t1 power r into t2 power s divided by r factorial into s factorial plus 1 minus rho square into summation over r is equal to zero to infinity, s is equal to zero to infinity mu r,s t1 power r plus 1 into t2 power s plus 1 divided by r factorial into s factorial Equating the coefficients of t1 power r minus 1 divided by r minus 1 factorial into t2 power s minus 1 divided by s minus 1 factorial on both sides and

simplifying we get, Mu r, s is equal to r plus s minus 1 into rho into mu r minus 1, s minus 1 plus r minus 1 into s minus 1 into 1 minus rho square into mu r minus 2, s minus 2.

5. Illustration

If X and Y are independent standard normal variates, obtain the moment generating function of X into Y. Let us find the moment generating function as follows
By definition, we have,M X into Y of t is equal to expectation of e power t into X into Y Is equal to double integral from minus infinity to infinity e power t into x into y into f of x y dx dy Since X and Y are independent standard normal variates, their joint pdf f(x,y) is given by, f of x, y is equal to 1 divided by 2 into phi into e power minus x square divided by 2 into e power minus y square divided by 2; where (x, y) lies between minus infinity to infinity.

Therefore, M X into Y of t is equal to 1 divided by 2 into phi into double integral from minus infinity to infinity e power minus half into x square minus 2 into t into x into y plus y square dx dy is equal to 1 divided by 2 into phi, double integral from minus infinity to infinity e power minus 1 divided by 2 into 1 minus t square, into x square divided by 1 by 1 minus t square minus 2 into t into x into y divided by 1 by square root of 1 minus t square into 1 by square root of 1 minus t square plus y square divided by 1 by 1 minus t square dx dy. If U and Y follows bivariate normal distribution with parameters zero, zero sigma 1 square, sigma 2 square and rho, then we have, Double integral from minus infinity to infinity, 1 divided by 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square, into x square divided by sigma 1 square minus 2 into rho into x into y divided by sigma 1 into sigma 2, plus y square divided by sigma 2 square, dx dy is equal to 1. Implies, double integral from minus infinity to infinity e power minus 1 divided by 2 into 1 minus rho square, into x square divided by sigma 1 square minus 2 into rho into x into y divided by sigma 1 into sigma 2 plus y square divided by sigma 2 square dx dy is equal to 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square. On comparison, we get, M X into Y of t is equal 1 divided by 2 into phi, into 2 into phi into 1 divided by square root of 1 minus t square into 1 divided by square root of 1 minus t square into square root of 1 minus t square Implies, M X into Y of t is equal to 1 minus t square whole power minus half.

Our learning in this session, where we have understood:

- The conditional distribution
- The coefficient of correlation
- The recurrence relation of central moments