Frequently Asked Questions

1. Define bivariate normal distribution with parameters μ_1, μ_2, σ_1, σ_2 and ρ?

Answer:

Random variable (X, Y) is said to follow bivariate normal distribution with parameters μ_1, μ_2, σ_1, σ_2 and ρ, if pdf is given by,

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)}$$

2. Write the marginal pdf of X in a bivariate normal distribution.

Answer:

Its marginal pdf of X is given by,

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$

3. Write the marginal pdf of Y in a bivariate normal distribution.

Answer:

Its marginal pdf of Y is given by,

$$f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2}$$

4. Derive the conditional distribution of X for fixed Y of bivariate normal distribution.

Answer:

Conditional distribution of X for fixed Y is given by,

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{\dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)}}{\dfrac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}(1-[1-\rho^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left((x-\mu_1) - \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left((x-\left\{\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right\}\right)^2}$$

Which is the probability function of a univariate normal distribution with mean and variance, which is given by,

$$E(X \mid Y) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), V(X \mid Y) = (1-\rho^2)\sigma_1^2$$

Hence, conditional distribution of X for fixed Y is also normal given by

$$(X \mid Y = y) \sim N\left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), (1-\rho^2)\sigma_1^2\right]$$

5. Write the mean and variance of conditional distribution of X for fixed Y in a bivariate normal distribution.

Answer:

Mean and variance of conditional distribution of X for fixed Y is given by,

$$E(X \mid Y) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), V(X \mid Y) = (1-\rho^2)\sigma_1^2$$

6. Obtain the conditional distribution of Y for fixed X of bivariate normal distribution.

Answer:

The conditional distribution of Y for fixed X is given by,

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x,y)}{f_x(x)} = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left(y-\left\{\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right\}\right)^2}$$

Thus, conditional distribution of Y for fixed X is also normal and given by,

$$(Y \mid X = x) \sim N\left[\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), (1-\rho^2)\sigma_2^2\right]$$

7. Write the mean and variance of conditional distribution of Y for fixed X in a bivariate normal distribution.

Answer:

Mean and variance of conditional distribution of Y for fixed X is given by,

$$E(Y \mid X) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), V(Y \mid X) = (1-\rho^2)\sigma_2^2$$

8. What happens when $\rho=0$ in conditional distribution of Y given X?

Answer:

For $\rho=0$, the conditional variance V(Y|X) is equal to the marginal variance σ_2^2 and the conditional mean of E(Y|X) is equal to the marginal mean μ_2 and the two variables become independent.

9. Show that if X and Y are standard normal variates with correlation coefficient ρ between them, then the correlation coefficient between X^2 and Y^2 is given by ρ^2.

Answer:

Given X and Y are two standard normal variates. Hence we have E(X)=E(Y)=0

$V(X)= E(X^2)= 1=V(Y)=E(Y^2)$

$$\therefore M_{X,Y}(t_1,t_2) = e^{\frac{1}{2}(t_1^2+2\rho t_1t_2+t_2^2)}$$

Now, consider the coefficient of correlation,

$$\rho(X^2,Y^2) = \frac{E(X_1^2X_2^2) - E(X_1^2)E(X_2^2)}{\sqrt{[E(X_1^4) - \{E(X_1^2)\}^2][E(X_2^4) - \{E(X_2^2)\}^2]}}$$

Where

$E(X_1^2X_2^2)$=Coefficient of $(t_1^2/2!).(t_2^2/2!)$ in $M(t_1,t_2)$

$=2\rho^2+1$

$E(X_1^4)$=Coefficient of $(t_1^4/4!)$ in $M(t_1,t_2)=3$

$E(X_2^4)$=Coefficient of $(t_2^4/4!)$ in $M(t_1,t_2)=3$

$$\therefore \rho(X^2,Y^2) = \frac{2\rho^2+1-(1\times 1)}{\sqrt{[3-1][3-1]}} = \rho^2$$

Therefore, the correlation coefficient between X^2 Y^2 is given by ρ^2.

10. Write moment generating function when (X,Y)~BVN(0, 0, 1, 1, ρ).

Answer: If (X,Y)~BVN(0, 0, 1, 1, ρ) then $M_{XY}(t_1,t_2) = e^{\frac{1}{2}\left(t_1^2+t_2^2+2\rho t_1 t_2\right)}$

11. If X and Y are standard normal variates with coefficient of correlation ρ, then show that, $Q=(X^2+2\rho XY+Y^2)/(1-\rho^2)$ is distributed like a chi-square, i.e. as that of the sum of the squares of standard normal variates.

Answer:

Consider the mgf of Q

$$M_Q(t) = E(e^{tQ}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tQ} f(x,y)\,dxdy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tQ} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}\,dxdy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tQ-\frac{Q}{2}}\,dxdy \; \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{Q}{2}(1-2t)}\,dxdy$$

Put x√(1-2t)=u and y√(1-2t)=v $\Rightarrow dx = \frac{du}{\sqrt{1-2t}}, dy = \frac{dv}{\sqrt{1-2t}}$

Also $Q = \frac{1}{1-\rho^2}\left[x^2 - 2\rho xy + y^2\right] = \frac{1}{1-\rho^2}\left[\frac{u^2-2\rho uv+v^2}{1-2t}\right]$

$$\therefore M_Q(t) = \frac{1}{2\pi(1-2t)\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(u^2-2\rho uv+v^2\right)}\,dudv = \frac{1}{(1-2t)}\times 1 = (1-2t)^{-1}$$

Which is the mgf of Chi-square variate with n(=2) degrees of freedom.

12. Show that for a bivariate normal distribution with $f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)}$, the moments obey the recurrence relation $\mu_{rs} = (r+s-1)\rho\mu_{r-1,s-1} + (r-1)(s-1)(1-\rho^2)\mu_{r-2,s-2}$.

Answer:

We know that mgf of above distribution is given by, $M = M(t_1,t_2) = e^{\frac{1}{2}\left(t_1^2+2\rho t_1 t_2+t_2^2\right)}$

$$\frac{\partial M}{\partial t_1} = M(t_1+\rho t_2), \frac{\partial M}{\partial t_2} = M(t_2+\rho t_1),$$

$$\frac{\partial^2 M}{\partial t_1 \partial t_2} = \frac{\partial}{\partial t_1}\left(\frac{\partial M}{\partial t_2}\right) = \frac{\partial}{\partial t_1}\left(M(t_2+\rho t_1)\right) = M\rho + (t_2+\rho t_1)(t_1+\rho t_2)M$$

$$\therefore \frac{\partial^2 M}{\partial t_1 \partial t_2} - \rho t_1 \frac{\partial M}{\partial t_1} - \rho t_2 \frac{\partial M}{\partial t_2}$$

$$= M\rho + (t_2+\rho t_1)(t_1+\rho t_2)M - \rho t_1(t_1+\rho t_2)M - \rho t_2(t_2+\rho t_1)M$$

$$= M(t_1 t_2 + \rho - \rho^2 t_1 t_2)$$

$$= M\rho + (1-\rho^2)M t_1 t_2$$

$$\therefore \frac{\partial^2 M}{\partial t_1 \partial t_2} = \rho t_1 \frac{\partial M}{\partial t_1} + \rho t_2 \frac{\partial M}{\partial t_2} + M\rho + (1-\rho^2)Mt_1t_2 ----(*)$$

But $M = e^{\frac{1}{2}\left(t_1^2 + 2\rho t_1 t_2 + t_2^2\right)} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \mu_{rs} \frac{t_1^r t_2^s}{r!s!}$

Therefore, (*) gives $\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \mu_{rs} \frac{t_1^{r-1} t_2^{s-1}}{(r-1)!(s-1)!}$

$$= \rho\sum_{r=1}^{\infty}\sum_{s=0}^{\infty} r\mu_{rs} \frac{t_1^r t_2^s}{r!s!} + \rho\sum_{r=0}^{\infty}\sum_{s=1}^{\infty} s\mu_{rs} \frac{t_1^r t_2^s}{r!s!} + \rho\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \mu_{rs} \frac{t_1^r t_2^s}{r!s!} + (1-\rho^2)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \mu_{rs} \frac{t_1^{r+1} t_2^{s+1}}{r!s!}$$

Equating the coefficient of $\frac{t_1^{r-1}}{(r-1)!}\frac{t_2^{s-1}}{(s-1)!}$ on both sides and simplifying we get, $\mu_{rs} = (r+s-1)\rho\mu_{r-1,s-1} + (r-1)(s-1)(1-\rho^2)\mu_{r-2,s-2}$

13. If moments obey the recurrence $\mu_{rs} = (r+s-1)\rho\mu_{r-1,s-1} + (r-1)(s-1)(1-\rho^2)\mu_{r-2,s-2}$ obtain μ_{22} and μ_{31}

Answer:

We have given $\mu_{rs} = (r+s-1)\rho\mu_{r-1,s-1} + (r-1)(s-1)(1-\rho^2)\mu_{r-2,s-2}$ and $\mu_{00}=1$ and $\mu_{11} = \rho\sigma_1\, \sigma_2 = \rho$

$\mu_{22} = (2+2-1)\rho\mu_{2-1,2-1} + (2-1)(2-1)(1-\rho^2)\mu_{2-2,2-2}$

$= 3\rho\mu_{1,1} + (1-\rho^2)\mu_{0,0} = 3\rho.\rho + (1-\rho^2).1 = 3\rho^2 + (1-\rho^2) = 1+2\rho^2$

14. If moments obey the recurrence $\mu_{rs} = (r+s-1)\rho\mu_{r-1,s-1} + (r-1)(s-1)(1-\rho^2)\mu_{r-2,s-2}$ obtain odd order central moment, i.e. when r+s is odd

Answer:

We have given $\mu_{rs} = (r+s-1)\rho\mu_{r-1,s-1} + (r-1)(s-1)(1-\rho^2)\mu_{r-2,s-2}$

We know that $\mu_{01}=\mu_{10}=0$

Also $\mu_{03}= \mu_{30}=0$

$\mu_{12}=2\rho\mu_{0,1}+0=0$

$\mu_{23}=4\rho\mu_{1,2}+1.2(1-\rho^2)\mu_{0,1}=0$

Similarly,-+

We get $\mu_{2,1}=0$ and $\mu_{3,2}=0$

If r+s is odd, so is (r-1)+(s-1), (r-2)+(s-2) and so on

and since $\mu_{03}= \mu_{30}$, $\mu_{12}= \mu_{21}$, and so on we finally get

$\mu_{rs}=0$ of r+s=0

15. If X and Y are independent standard normal variates, obtain the mgf of XY.

Answer:

We have by definition $M_{XY}(t) = E(e^{tXY}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{txy} f(x,y)dxdy$

Since X and Y are independent standard normal variates, their joint pdf f(x,y) is given by, $f(x,y) = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}; -\infty < (x,y) < \infty$

$$\therefore M_{XY(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2 - 2txy + y^2\right)} dxdy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-t^2)}\left(\frac{x^2}{1/(1-t^2)} - \frac{2txy}{(1/\sqrt{(1-t^2)})(1/\sqrt{(1-t^2)})} + \frac{y^2}{1/(1-t^2)}\right)} dxdy$$

If (U,V)~BVN(0,0,σ_1^2, σ_2^2,ρ) then we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2)}\right)} dxdy = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2)}\right)} dxdy = 2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}$$

On comparison we get,

$$\therefore M_{XY}(t) = \frac{1}{2\pi} 2\pi \frac{1}{\sqrt{1-t^2}} . \frac{1}{\sqrt{1-t^2}} . \sqrt{1-t^2} \Rightarrow M_{XY}(t) = (1-t^2)^{-1/2}$$