Statistics
Bivariate Normal Distribution (Part-1)

1. Introduction

Welcome to the series of E-learning modules on Bivariate Normal distribution.

The bivariate normal distribution is the generalisation of a normal distribution for a single variate. Let X and Y be the two normally correlated variables with correlation coefficient of rho and Expectation of X is equal to mu 1, Variance of X is equal to sigma 1 square; Expectation of Y is equal to mu 2, Variance of Y is equal to sigma 2 square. In deriving the bivariate normal distribution, we make the following three assumptions:

- The regression of Y on X is linear. Since the mean of each array is on the line of regression Y is equal to rho into sigma 2 divided by sigma 1 into X, the mean or expected value of Y is rho into sigma 2 divided by sigma 1 into X, for different values of X.

- The arrays are homoscedastic, that is variance in each array is same. The common variance of estimate of Y in each array is given by sigma 2 square into 1 minus rho square, rho being the correlation coefficient between variables X and y and is independent of X.

- The distribution of Y in different arrays is normal. Suppose that one of the variates, say X, is distributed normally with mean 0 and standard deviation $\sigma 1$ so that the probability that a random value of X will fall in the small interval dx is g of x dx is equal to 1 divided by sigma 1 into square root of 2 into phi into e power minus x square divided by 2 into sigma 1 square.

The probability that a value of Y, taken at random in an assigned vertical array will fall in the interval dy is h of y given x dy is equal to 1 divided by sigma 2 into square root of 2 into phi into 1 minus rho square into e power minus 1 divided by 2 into sigma 2 square into 1 minus rho square into y minus rho into sigma 2 divided by sigma 1 into x the whole square

The joint probability differential of X and Y is given by dP of x, y is equal to g of x into h of y given x dx, dy Is equal to 1 divided by 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square into e power minus x square divided by 2 into sigma 1 square, into e power minus 1 divided by 2 into sigma 2 square into 1 minus rho square, into y minus rho into sigma 2 divided by sigma 1 into x the whole square dx, dy
Is equal to 1 divided by 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square into x square divided by sigma 1 square minus 2 into rho into x into y divided by sigma 1 into sigma 2, plus y square divided by sigma 2 square, the whole square dx, dy

Shifting the origin to mu 1 and mu 2, we get, f XY of x, y is equal to 1 divided by 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square into x minus mu 1 the whole square divided by sigma 1 square minus 2 into rho into x minus mu 1 into y minus mu 2 whole divided by sigma 1 into sigma 2 plus y minus mu 2 whole square divided by sigma 2 square, Where, mu 1, mu 2, sigma 1, greater than zero, sigma 2, greater than zero, and rho, lies between minus 1 and 1 are the five parameters of the distribution.

2. Normal Correlation Surface

Following figure shows the normal correlation surface.

If X Y of (x, y) gives the density function of the bivariate normal distribution. The variables X and Y are said to be normally correlated and the surface Z is equal to f of (x, y) is known as the normal correlation surface. The nature of the normal correlation surface is indicated in the above diagram.

Consider the following remarks:

- The vector (X,Y) dash following the joint probability density function f of x, y will be abbreviated as (X,Y) follows Normal (mu 1, mu 2, sigma 1 square, sigma 2 square, rho) or Bivariate Normal (mu 1, mu 2, sigma 1 square, sigma 2 square, rho). If in particular, mu 1 is equal to mu 2 is equal to zero and sigma 1 is equal to 1, sigma 2 is equal to 1 then (X,Y) follows Bivariate Normal (zero, zero, 1, 1, rho)
- The curve z is equal to f of (x, y) which is the equation of a surface in three dimension is called as the ‘Normal Correlation Surface’.

3. Moment Generating Function (mgf)

Now, let us find the moment generating function of bivariate normal distribution.
Let X follows Bivariate Normal (mu 1, mu 2, sigma 1 square, sigma 2 square, rho). By definition,
M XY of (t_1, t_2) is equal to Expectation of e power t into X plus t2 into Y
Is equal to double integral from minus infinity to infinity e power t1 into x plus t2 into y f of x, y dx dy
Put u is equal to x minus mu 1 divided by sigma 1 and v is equal to y minus mu 2 divided by sigma 2, where minus infinity less than u and v less than infinity.
That is x is equal to sigma 1 into u plus mu 1, y is equal to sigma 2 into v plus mu 2 implies modulus of Jacobean j is equal to sigma 1 into sigma 2.
Therefore, M XY of t1, t2 is equal to e power t1 into mu 1 plus t2 into mu2 divided by 2 into phi into square root of 1 minus rho square into double integral from minus infinity to infinity e power t1 into sigma 1 into u plus t2 into sigma 2 into v minus 1 divided by 2 into 1 minus rho square into u square into 2 into rho into u into v plus v square du, dv

Is equal to e power t1 into mu 1 plus t2 into mu 2 divided by 2 into phi into square root of 1 minus rho square into double integral from minus infinity to infinity e power t1 into minus sigma 1 into u plus t2 into sigma 2 into v minus 1 divided by 2 into 1 minus rho square into u square minus 2 into rho into u into v plus v square du, dv.

Is equal to e power t1 into mu 1 plus t2 into mu 2 divided by 2 into phi into square root of 1 minus rho square into double integral from minus infinity to infinity e power minus 1 divided by 2 into 1 minus rho square into u square minus 2 into rho into u into v plus v square minus 2 into 1 minus rho square into t1 into sigma 1 into u plus t2 into sigma 2 into v du, dv. Name this equation as star.

We have u square minus 2 into rho into u into v plus v square minus 2 into 1 minus rho square into t1 into sigma 1 into u plus t2 into sigma 2 into v

Is equal to u minus rho into v minus 1 minus rho square into t1 into sigma 1 the whole square plus 1 minus rho square into v minus rho into t1 into sigma 1 minus t2 into sigma 2 square whole square minus t1 square into sigma 1 square minus t2 square into sigma 2 square minus 2 into rho into t1 into t2 into sigma 1 into sigma 2

By taking u minus rho into v minus 1 minus rho square into t1 into sigma 1 is equal to omega into 1 minus rho square whole power half.

And v minus rho into t1 into sigma 1 minus t2 into sigma 2 is equal to z Implies, du, dv is equal to 1 minus rho square whole power half dw, dz

And using this in star we get, M XY of t1, t2 is equal to e power t1 into mu 1 plus t2 into mu2 plus half into t1 square into sigma 1 square plus t2 square into sigma 2 square plus 2 into rho into t1 into t2 into sigma 1 into sigma 2 into integral from minus infinity to infinity 1 divided by square root of 2 into phi into e power minus omega square divided by 2 d omega into integral from minus infinity to infinity 1 divided by square root of 2 into phi into e power minus z square by 2 dz
Is equal to e power t1 into mu 1 plus t2 into mu2 plus half into t1 square into sigma 1 square plus t2 square into sigma 2 square plus 2 into rho into t1 into t2 into sigma 1 into sigma 2

In particular if X, Y follows bivariate normal distribution with parameters zero, zero, 1, 1 and rho, then M XY of t1, t2 is equal to e power half into t1 square plus t2 square plus 2 into rho into t1 into t2.

4. Conditions for Independence of the Variables in Bivariate Normal Distribution

Consider the following theorem.

Let X, Y follows bivariate normal distribution with parameters mu 1, mu 2, sigma 1 square, sigma 2 square, rho. Then, X and Y are independent if and only if rho is equal to zero.

Let us consider two cases,

- If X Y follows bivariate normal distribution with parameters mu 1, mu 2, sigma 1 square, sigma 2 square, rho and rho is equal to zero, then X and Y are independent.
  If rho is equal to zero, then f of x, y is equal to 1 divided by 2 into phi into sigma 1 into sigma 2 into e power minus half into x minus mu 1 divided by sigma 1 whole square into e power minus half into y minus mu 2 divided by sigma 2 the whole square
  Is equal to 1 divided by square root of 2 into phi into sigma 1 into e power minus half into x minus mu 1 divided by sigma 1 the whole square into 1 divided by square root of 2 into phi, into sigma 2 into e power minus half into y minus mu 2 divided by sigma 2 the whole square.
  Implies, f of x, y is equal to f1 of x into f2 of y.
  Therefore, X and Y are independent.

- Conversely, if X and Y are independent, then rho is equal to zero.
  We know that if X and Y are independent, then covariance of X, Y is equal to zero.
  Implies rho is equal to Covariance of X, Y divided by sigma 1 into sigma 2 is equal to zero.

X, Y possesses a bivariate normal distribution if and only if every linear combination of X and Y namely a into X plus b into Y, where a is not equal to zero and b is not equal to zero is a normal variate.
Solution:

Let (X, Y) follows bivariate normal distribution with parameters mu 1, mu 2, sigma 1 square, sigma 2 square, rho, then we shall prove that a into X plus b into Y, where a not equal to zero and b is not equal to zero is a normal variate.
Since (X,Y) has a bivariate normal distribution we have, M XY of t1, t2 is equal to Expectation of e power t1 into X plus t2 into Y
Is equal to e power t1 into mu 1 plus t2 into mu 2 plus half into t1 square into sigma 1 square plus t2 square into sigma 2 square plus 2 into rho into t1 into t2 into sigma 1 into sigma 2

The moment generating function of Z is equal to a into X into b into Y is given by
MZ of t is equal to expectation of e power t into Z
Is equal to e power t into a into X plus b into Y
Is equal to e power a into t into X plus b into t into Y
Is equal e power t into a into mu 1 plus b into mu 2 plus t square by 2 into sigma 1 square plus sigma 2 square plus 2 into rho into sigma 1 into sigma 2, which is the moment generating function of normal distribution with parameters mu is equal to a into mu 1 plus b into mu 2 and sigma square is equal to a square into sigma 1 square plus 2 into rho into a into b into sigma 1 into sigma 2 plus b square into sigma 2 square.
Hence, by uniqueness theorem of moment generating function, Z is equal to a into X plus b into Y following normal mu and sigma square, where mu and sigma square are defined as above.

- Conversely, let Z is equal to a into X plus b into Y, a not equal to zero, b not equal to zero be a normal variate. Then we have prove that (X,Y) has a bivariate normal distribution.

Let Z is equal to a into X plus b into Y follows Normal mu, sigma square, where mu is equal to expectation of Z is equal to a into mu X plus b into mu Y and
Sigma square is equal to a square into sigma x square plus 2 into a into b into rho into sigma x into sigma y plus b square into sigma y square.

M Z of t is equal to e power t into mu plus t square into sigma square by 2
Is equal to e power t into a into mu x plus b into mu y plus, t square into a square into sigma x square plus 2 into a into b into rho sigma x into sigma y into b square into sigma y square by 2.

Is equal to e power t1 into mu x plus t2 into mu y plus t1 square into sigma x square plus 2 into rho into t1 into t2 into sigma x into sigma y plus t2 square into sigma y square by 2. Where t1 is equal to a into t and t2 is equal to b into t.

Above expression is the mgf of bivariate normal distribution with parameters mu 1, mu 2, sigma 1 square, sigma 2 square and rho.

Hence, by uniqueness theorem of mgf, (X, Y) follows bivariate normal (mu 1, mu 2, sigma 1 square, sigma 2 square and rho).

5. Marginal Distribution

Now, let us find the marginal distributions of bivariate normal distribution.
The marginal distribution of random variable X is given by, fX of x is equal to integral from minus infinity to infinity f XY of x, y dy. Put y minus mu 2 divided by sigma 2 is equal to u, then dy is equal to sigma 2 into du. Therefore, fX of x is equal to 1 divided by 2 into phi into sigma 1 into sigma 2 into square root of 1 minus rho square into integral from minus infinity to infinity e power minus 1 divided by 2 into 1 minus rho square into x minus mu 1 the whole square divided by sigma 1 square minus 2 into rho into u into x minus mu 1 divided by sigma 1 plus u square, into sigma 2, du

Is equal to 1 divided by 2 into phi into sigma 1 square root of 1 minus rho square into e power minus x minus mu 1 whole square divided by 2 into sigma 1 square into integral from minus infinity to infinity e power minus 1 divided by 2 into 1 minus rho square, into u minus rho into x minus mu 1 divided by sigma 1, the whole square, du

Put 1 divided by square root of 1 minus rho square, into u minus rho into x minus mu 1 divided by sigma 1 is equal to t then du is equal to square root of 1 minus rho square, dt.
Therefore, fX of x is equal to 1 divided by 2 into phi into sigma 1, into e power minus half into x minus mu 1 divided by sigma 1 the whole square, into integral from minus infinity to infinity e power minus t square divided by 2 dt. Is equal to 1 divided by 2 into phi into sigma 1, into e power minus half into x minus mu 1 divided by sigma 1 the whole square, into square root of 2 into phi

Is equal to 1 divided by sigma 1 into square root of 2 into phi, e power minus half into x minus mu 1 divided by sigma 1 the whole square.
Similarly we get, fY of y is equal to 1 divided by sigma 2 into square root of 2 into phi into e power minus half into x minus mu 2 divided by sigma 2 the whole square.

Hence, X follows Normal distribution with parameters mu 1 and sigma 1 square and Y follows normal distribution with parameters mu 2 and sigma 2 square.

Consider the following remark: We have proved that if X, Y follows Bivariate Normal distribution with parameters mu 1, mu 2, sigma 1 square, sigma 2 square and rho, then the marginal probability density functions of X and Y are also normal. However, the converse is not true. That is we may have joint probability density function of f of x, y of X, Y which is not normal but the marginal probability density functions may be normal, which is shown in the following illustration.

Consider the joint distribution of X and Y, which is given by, f of x, y is equal to half into 1 divided by 2 into phi into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square, into x square minus 2 into rho into x into y plus y square, plus 1 divided by 2 into phi into square root of 1 minus rho square into e power minus 1 divided by 2 into 1 minus rho square, into x square plus 2 into rho into x into y plus y square. Is equal to half into f 1 of x plus f 2 of x, where minus infinity less than x and y less than infinity.

Observe that f1 of x, y is the probability density function of Bivariate Normal distribution with parameters (zero, zero, 1, 1, and rho) and f2 of x, y is the probability density function of Bivariate normal distribution with parameters (zero, zero, 1, 1, and minus rho). It can be easily verified that f of (x, y) is the joint probability density function of (X, Y), obviously f if (x, y) is not the probability density function of bivariate normal distribution.

Here’s a summary of our learning in this session, where we have understood:

- The derivation of probability function of bivariate normal distribution
- The normal correlation surface
- The moment generating function
- The conditions for independence of the variables in bivariate normal distribution
- The marginal distribution of bivariate normal distribution