Frequently Asked Questions

1. How bivariate normal distribution is obtained?

Answer:

The bivariate normal distribution is the generalisation of a normal distribution for a single variate. Let X and Y be two normally correlated variables with correlation coefficient of ρ and $E(X)=\mu_1$, $Var(X)=\sigma_1^2$; $E(Y)=\mu_2$, $Var(Y)=\sigma_2^2$, the (X, Y) is bivariate normal.

2. Write the assumptions made in deriving the bivariate normal distribution.

Answer:

- The regression of Y on X is linear. Since the mean of each array is on the line of regression $Y=\rho(\sigma_2/\sigma_1)X$, the mean or expected value of Y is $\rho(\sigma_2/\sigma_1)X$, for different values of X.
- The arrays are homoscedastic, i.e., variance in each array is same. The common variance of estimate of Y in each array is then given by $\sigma_2^2(1-\rho^2)$, ρ being the correlation coefficient between variables X and y and is independent of X.
- The distribution of Y in different arrays is normal.

3. Derive the pdf of bivariate normal distribution.

Answer:

Let us do the following assumptions:

i. The regression of Y on X is linear. Since the mean of each array is on the line of regression $Y=\rho(\sigma_2/\sigma_1)X$, the mean or expected value of Y is $\rho(\sigma_2/\sigma_1)X$, for different values of X.

ii. The arrays are homoscedastic, i.e., variance in each array is same. The common variance of estimate of Y in each array is then given by $\sigma_2^2(1-\rho^2)$, ρ being the correlation coefficient between variables X and y and is independent of X.

iii. The distribution of Y in different arrays is normal. Suppose that one of the variates, say X, is distributed normally with mean 0 and standard deviation σ_1 so that the probability that a random value of X will fall in the small interval dx is

$$g(x)dx = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}}$$

The probability that a value of Y, taken at random in an assigned vertical array will fall in the interval dy is $h(y \mid x)dy = \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left(y-\rho\frac{\sigma_2}{\sigma_1}x\right)^2}$

The joint probability differential of X and Y is given by

$dP(x,y)=g(x)h(y|x)\,dx\,dy$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{x^2}{2\sigma_{2=1}^2}} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left(y-\rho\frac{\sigma_2}{\sigma_1}x\right)^2} dxdy$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2\rho xy}{\sigma_1\sigma_2}x+\frac{y^2}{\sigma_2^2}\right)^2} dxdy$$

Shifting the origin to (μ_1,μ_2),we get

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)};$$

Where μ_1, μ_2, $\sigma_1(>0)$, $\sigma_2(>0)$ and $\rho(-1<\rho<1)$ are the five parameters of the distribution.

4. Write the different parameters of bivariate normal distribution with their range.

Answer:

Different parameters of normal distribution are μ_1, μ_2, σ_1, σ_2 and ρ. Their ranges are,

$-\infty<\mu_1<\infty$

$-\infty<\mu_2-\infty$

$\sigma_1>0$,

$\sigma_2>0$ and

$-1<\rho<1$

5. What is normal correlation surface?

Answer:

The curve z=f(x,y) which is the equation of a surface in three dimension is called the 'Normal Correlation Surface'.

6. Obtain moment generating function of bivariate normal distribution.

Answer:

Let X~ BVN(μ_1, μ_2, σ_1^2, σ_2^2, ρ). By definition,

$M_{XY}(t_1, t_2) = E\left[e^{t_1X+t_2Y}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dxdy$

Put $u = \frac{x-\mu_1}{\sigma_1}$ and $v = \frac{y-\mu_2}{\sigma_2}$, $-\infty < (u,v) < \infty$

i.e., $x=\sigma_1 u+\mu_1$, $y=\sigma_2 v+\mu_2 \Rightarrow |J|=\sigma_1\sigma_2$

$\therefore M_{XY}(t_1, t_2) = \frac{e^{t_1\mu_1+t_2\mu_2}}{2\pi\sqrt{1-\rho^2}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1\sigma_1 u+t_2\sigma_2 v-\frac{1}{2(1-\rho^2)}\{u^2-2\rho uv+v^2\}} dudv$

$= \frac{e^{t_1\mu_1+t_2\mu_2}}{2\pi\sqrt{1-\rho^2}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left\{\left(u^2-2\rho uv+v^2\right)-2(1-\rho^2)(t_1\sigma_1 u+t_2\sigma_2 v)\right\}} dudv$ -------(*)

We have $(u^2-2\rho uv+v^2)-2(1-\rho^2)(t_1\sigma_1u+t_2\sigma_2v)$

$=\{(u-\rho v)-(1-\rho^2)t_1\sigma_1\}^2+(1-\rho^2)\{(v-\rho t_1\sigma_1-t_2\sigma_2)^2-t_1^2\sigma_1^2-t_2^2\sigma_2^2-2\rho t_1t_2\sigma_1\sigma_2\}$

By taking $u-\rho v-(1-\rho^2)t_1\sigma_1=\omega(1-\rho^2)^{1/2}$

And $v-\rho t_1\sigma_1-t_2\sigma_2=z$

$\Rightarrow du\,dv = (1-\rho^2)^{1/2}dw\,dz$

And using in *, we get

$$M_{XY}(t_1,t_2) = e^{t_1\mu_1+t_2\mu_2+\frac{1}{2}\left(t_1^2\sigma_1^2+t_2^2\sigma_2^2+2\rho t_1t_2\sigma_1\sigma_2\right)} \times \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\varpi^2/2}d\varpi \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz$$

$$= e^{t_1\mu_1+t_2\mu_2+\frac{1}{2}\left(t_1^2\sigma_1^2+t_2^2\sigma_2^2+2\rho t_1t_2\sigma_1\sigma_2\right)}$$

7. Write moment generating function of BVN(0, 0, 1, 1, ρ)

Answer:

We know that, in general $M_{XY}(t_1, t_2) = e^{t_1\mu_1+t_2\mu_2+\frac{1}{2}\left(t_1^2\sigma_1^2+t_2^2\sigma_2^2+2\rho t_1t_2\sigma_1\sigma_2\right)}$

In particular if (X,Y)~BVN(0, 0, 1, 1, ρ) then

$$M_{XY}(t_1,t_2) = e^{\frac{1}{2}\left(t_1^2+t_2^2+2\rho t_1t_2\right)}$$

8. If (X, Y)~BVN(μ_1 ,μ_2, σ_1^2, σ_2^2, ρ), then X and Y are independent if and only if ρ=0.

Answer:

Since we have if and only if condition, we consider the following two parts.

a. If (X, Y)~BVN(μ_1 ,μ_2, σ_1^2, σ_2^2, ρ) and ρ=0, then X and Y are independent.

If ρ=0 then

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} . e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2} = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} . \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$

$\Rightarrow f(x,y)=f_1(x).f_2(y)$

∴ X and Y are independent.

b. Conversely, if X and Y are independent, then ρ=0

If X and y are independent, then Cov(X,Y)=0

$$\Rightarrow \rho = \frac{Cov(X, Y)}{\sigma_1\sigma_2} = 0$$

9. Show that (X, Y) possesses a bivariate normal distribution if and only if every linear combination of X and Y namely, aX+bY, a≠0, b≠0, is a normal variate.

Answer:

Since we have if and only if condition, we consider the following two parts.

a. Let (X, Y)~BVN(μ_1 ,μ_2, σ_1^2, σ_2^2, ρ), then we shall prove that aX+bY, a≠0, b≠0 is a normal variate.

Since (X,Y) has a bivariate normal distribution we have,

$M_{X,Y}(t_1, t_2)=E(e^{t1X+t2Y}) = e^{t_1\mu_1+t_2\mu_2+\frac{1}{2}\left(t_1^2\sigma_1^2+t_2^2\sigma_2^2+2\rho t_1t_2\sigma_1\sigma_2\right)}$

The mgf of Z=aX+bY is given by,

$M_Z(t)=E(e^{tZ})=E[e^{t(aX+bY)}]=E(e^{atX+btY})$

$$= e^{t(a\mu_1+b\mu_2)+\frac{t^2}{2}\left(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2\right)},$$

Which is the mgf normal distribution with parameters $\mu=a\mu_1+b\mu_2$, $\sigma^2=a^2\sigma_1^2+2\rho ab\sigma_1\sigma_2+b^2\sigma_2^2$

Hence by uniqueness theorem of mgf, Z=aX+bY~N(μ, σ^2), where μ and σ^2 are defined as above.

b. Conversely, let Z=aX+bY, a≠0, b≠0 be a normal variate. Then we have prove that (X,Y) has a bivariate normal distribution.

Let Z=aX+bY ~ N(μ, σ^2), where

$\mu=E(Z)=a\mu_x+b\mu_y$ and

$\sigma^2=a^2\sigma_x^2+2ab\rho\sigma_x\sigma_y+b^2\sigma_y^2$

$$M_Z(t) = e^{t\mu+t^2\sigma^2/2}$$

$$= e^{t(a\mu_x+b\mu_y)+t^2(a^2\sigma_x^2+2ab\rho\sigma_x\sigma y+b^2\sigma_y^2)/2}$$

$$= e^{t_1\mu_x+t_2\mu_y+(t_1^2\sigma_x^2+2\rho t_1t_2\sigma_x\sigma y+t_2^2\sigma_y^2)/2}$$

t_1=at and t_2=bt. Above expression is the mgf of bivariate normal distribution with parameters μ_1 ,μ_2, σ_1^2, σ_2^2, ρ

Hence by uniqueness theorem of mgf,

(X, Y)~BVN(μ_1 ,μ_2, σ_1^2, σ_2^2, ρ),

10. Obtain marginal distribution of X of Bivariate normal distribution.

Answer:

The marginal distribution of random variable X is given by,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

Put $\frac{y-\mu_2}{\sigma_2} = u$, then dy=$\sigma_2$ du. Therefore

$$f_X(x) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho u\left(\frac{x-\mu_1}{\sigma_1}\right) + u^2\right]}\sigma_2 du$$

$$= \frac{1}{2\pi\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2} du$$

Put $\frac{1}{\sqrt{(1-\rho^2)}}\left[u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right] = t$, then du= $= \sqrt{(1-\rho^2)}dt$

$$\therefore f_X(x) = \frac{1}{2\pi\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \frac{1}{2\pi\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \sqrt{2\pi} = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$

11. Obtain marginal distribution of Y.

Answer:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

Put $\frac{x-\mu_1}{\sigma_1} = v$, then dv=$\sigma_1$ dv. Therefore

$$f_y(y) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[v^2 + \frac{(x-\mu_2)^2}{\sigma_2^2} - 2\rho v\left(\frac{x-\mu_2}{\sigma_2}\right) + \right]}\sigma_1 dv$$

$$= \frac{1}{2\pi\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[v-\rho\left(\frac{x-\mu_2}{\sigma_2}\right)\right]^2} dv$$

Put $\frac{1}{\sqrt{(1-\rho^2)}}\left[v-\rho\left(\frac{x-\mu_2}{\sigma_2}\right)\right] = t$, then dv= $= \sqrt{(1-\rho^2)}dt$

$$\therefore f_Y(y) = \frac{1}{2\pi\sigma_2} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \frac{1}{2\pi\sigma_2} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2} \sqrt{2\pi} = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2}$$

12. Prove that if (X,Y)~BVN(μ1 ,μ2, σ12, σ22, ρ), then the marginal pdf's of X and Y are also normal.

Answer:

The marginal distribution of random variable X is given by,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

Put $\dfrac{y-\mu_2}{\sigma_2} = u$, then dy=$\sigma_2$ du. Therefore

$$f_X(x) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho u\left(\frac{x-\mu_1}{\sigma_1}\right) + u^2\right]}\sigma_2 du$$

$$= \frac{1}{2\pi\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2} du$$

Put $\dfrac{1}{\sqrt{(1-\rho^2)}}\left[u-\rho\left(\dfrac{x-\mu_1}{\sigma_1}\right)\right] = t$, then du= $= \sqrt{(1-\rho^2)}dt$

$$\therefore f_X(x) = \frac{1}{2\pi\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt = \frac{1}{2\pi\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\sqrt{2\pi} = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

Put $\dfrac{x-\mu_1}{\sigma_1} = v$, then dv=$\sigma_1$ dv. Therefore

$$f_y(y) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[v^2 + \frac{(x-\mu_2)^2}{\sigma_2^2} - 2\rho v\left(\frac{x-\mu_2}{\sigma_2}\right) + \right]}\sigma_1 dv$$

$$= \frac{1}{2\pi\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[v-\rho\left(\frac{x-\mu_2}{\sigma_2}\right)\right]^2} dv$$

Put $\dfrac{1}{\sqrt{(1-\rho^2)}}\left[v-\rho\left(\dfrac{x-\mu_2}{\sigma_2}\right)\right] = t$, then dv= $= \sqrt{(1-\rho^2)}dt$

$$\therefore f_Y(y) = \frac{1}{2\pi\sigma_2} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt = \frac{1}{2\pi\sigma_2} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2}\sqrt{2\pi} = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2}$$

13. Whether the joint pdf of two normal variates is bivariate normal distribution?

Answer:

We may have joint pdf of f(x, y) of (X,Y) which is not normal but the marginal pdfs may be normal.

14. Illustrate with an example that joint pdf of f(x, y) of (X, Y) which is not normal but the marginal pdfs may be normal.

Answer:

We may have joint pdf of f(x, y) of (X, Y) which is not normal but the marginal pdfs may be normal. Consider the joint distribution of X and Y, which is given by

$$f(x,y) = \frac{1}{2}\left[\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)} + \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2+2\rho xy+y^2)}\right]$$

$$= \frac{1}{2}\left[f_1(x) + f_2(x)\right], -\infty < (x,y) < \infty$$

Observe that $f_1(x, y)$ is the pdf of BVN(0,0,1,1,ρ) and $f_2(x, y)$ is the pdf of BVN(0,0,1,1,-ρ). It can be easily verified that f(x, y) is the joint pdf of (X,Y), obviously f(x,y) is not the pdf of bivariate normal distribution.

15. Show that (X, Y) possesses a bivariate normal distribution if every linear combination of X and Y namely, aX+bY, a≠0, b≠0, is a normal variate.

Answer:

Let (X, Y)~BVN(μ_1, μ_2, σ_1^2, σ_2^2, ρ), then we shall prove that aX+bY, a≠0, b≠0 is a normal variate.

Since (X,Y) has a bivariate normal distribution we have,

$M_{X,Y}(t_1, t_2)=E(e^{t1X+t2Y}) = e^{t_1\mu_1+t_2\mu_2+\frac{1}{2}\left(t_1^2\sigma_1^2+t_2^2\sigma_2^2+2\rho t_1 t_2\sigma_1\sigma_2\right)}$

The mgf of Z=aX+bY is given by,

$M_Z(t)=E(e^{tZ})=E[e^{t(aX+bY)}]=E(e^{atX+btY})$

$= e^{t(a\mu_1+b\mu_2)+\frac{t^2}{2}\left(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2\right)}$,

Which is the mgf normal distribution with parameters $\mu=a\mu_1+b\mu_2$, $\sigma^2=a^2\sigma_1^2+2\rho ab\sigma_1\sigma_2+b^2\sigma_2^2$

Hence, by uniqueness theorem of mgf, Z=aX+bY~N(μ, σ^2), where μ and σ^2 are defined as above.