Statistics

Computations of Expectations, Moments and Moment Generating Functions

1. Introduction

Welcome to the series of E-learning modules on Computations of Expectations, Moments and Moment Generating Functions.

By the end of this session, you will be able to:

- Relate real life situations to a particular distribution and find different constants of the distribution
- To find mean and variance of a particular distribution
- To compute probabilities of different situations

2. Practical - 1

For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. What is the expectation and standard deviation of the distribution?

Solution:
We know that mu 1 dash is the first moment about the point X is equal to A, the Expectation; the mean is given by, A plus mu 1 dash.
We are given mu 1 dash about point x is equal to 10, is equal to 40 Implies Mean is equal 10 plus 40 is equal to 50.

Also, mu 4 dash about the point X is equal 50 is equal to 48. Since mean is also 50, μ 4 dash is equal to mu 4 is equal to 48 But for a normal distribution with standard deviation sigma, Mu 4 is equal to 3 into sigma power 4 is equal to 48 Implies sigma power 4 is equal to 16 or sigma is equal to 2.

Consider the 2^{nd} exercise
X is normally distributed and the mean of X is 12 and standard deviation is 4. Find probability of the following:
i. (a) X greater than or equal to 20
(b)X less than or equal to 5
(c)zero less than or equal to x less than or equal to 12

ii. Find x when Probability of (X greater than x) is equal to zero point 2, 4
iii. Find x1 and x2 when Probability of (x1 less than X less than x2) is equal to zero point 5 and Probability of (X greater than x1) is equal to zero point 2, 5.

Solution: We have mu is equal to 12 and sigma is equal to 4. That is X follows Normal distribution with parameters (12, and 16) and the corresponding standard normal variate is Z is equal to X minus mu divided by sigma is equal to X minus 12 divided by 4.

(i) We can find all the probabilities from standard normal tables. Hence, every time we convert the original variable X into standard normal variable Z.
a. Probability of (X greater than or equal to 20)
Is equal to Probability of [Z greater than or equal to (20 minus 12) divided by 4]
Is equal to Probability of (Z greater than or equal to 2)
Is equal to zero point zero 2, 2, 8
b. Probability of (Z less than or equal to 5)
is equal to Probability of [Z less than or equal to (5 minus 12) divided by 4]
is equal to Probability of (Z less than or equal to minus 1 point 7, 5)
is equal to zero point zero 4 zero, zero 6
c. Probability of (zero less than or equal to X less than or equal to 12)
Is equal to Probability of [(zero minus 12) divided by 4 less than or equal to Z less than or equal to (12 minus 12) divided by 4]
Is equal to Probability of (minus 3 less than or equal to Z less than or equal to zero)
Is equal to zero point 4, 9, 8, 6, 5

(ii) Given Probability of (X greater than x) is equal to zero point 2, 4
Implies Probability of (Z greater than z) is equal to zero point 2, 4, where z is equal to z minus 12 divided by 4.
The shaded area in the figure shows the given probability.

From the above curve, we have z is equal to zero point 7, 1 That is x minus 12 divided by 4 is equal to zero point 7, 1 Implies x is equal to 12 plus 4 into zero point 7,1 Is equal to 14. 84.

iii. we are given Probability of x1 less than x less than x2 is equal to zero point 5 and Probability of X greater than x1 is equal to zero point 2, 5. That is, Probability of z1 less than Z less than z2 is equal to zero point 5 and Probability of Z greater than z1 is equal to zero point 2, 5. Where, z1 is equal to x1 minus 12 divided by 4 and z2 is equal to x2 minus 12 divided by 4. The point's z_1 and z_2 are located as shown in the following curve.

From the curve, Probability of (Z greater than z_1) is equal to Probability of (Z less than z_2) or Probability of (z less than minus z_1) is equal to zero point 2, 5, as we know that normal curve is symmetric about zero. Hence, from standard normal tables, z_1 is equal to zero point 6, 7 and z2 is equal to minus z1 is equal to minus zero point 6, 7 or X1 is equal to 12 plus (4) into (zero point 6, 7) is equal to 14 point 6, 8 and X2 is equal to 12 plus (4) into (minus zero point 6, 7) is equal to 9 point 3, 2.

3. Practical - 2

In a distribution exactly normal, 10 point zero 3 percent of the items are under 25 kilogram weight and 89 point 9, 7 percent of the items are under 70 kilogram weight. What are the mean and standard deviation of the distribution?

Solution:
Let X denote the weight of the items in kg. If X follows normal distribution with parameters (mu and sigma square), then we are given Probability of (X less than 25) is equal to zero point 1 zero, zero, 3 and Probability of (X less than 70) is equal to zero point 8, 9, 9, 7. The points X is equal to 25 and X is equal

to 70 are located as below.

Since the value X is equal to 25 is located to left of the ordinate X is equal to mu (because left to this value probability is less than zero point 5), the corresponding value of Z is negative. Z is equal to 25 minus mu divided by sigma is equal to minus z1. Name it as equation number (1) Similarly, the value X is equal to 70 is located to right of the ordinate X is equal to μ (because left to this value probability is greater than zero point 5), the corresponding value of Z is positive. Z is equal to 70 minus mu divided by sigma is equal to z_2. Name it as equation number (2) Let us draw the diagram for indicating the probabilities.

From the diagram, it is obvious that, Probability of (Z less than minus z1) is equal to zero point 1, zero, zero 3 and Probability of (Z less than z2) is equal to zero point 8, 9, 9, 7 Using standard normal tables we get, z_2 is equal to 1 point 2, 8 and z_1 is equal to 1 point 2, 8 Substituting the values of z1 and z2 in (1) and (2), we get, 25 minus mu divided by sigma is equal to minus 1 point 2, 8 implies, 25 minus mu is equal to minus 1 point 2, 8 into sigma. Name it as equation number (3) And 70 minus mu divided by sigma is equal to 1 point 2, 8 implies, 70 minus mu is equal to 1 point 2, 8 into sigma. Name it as equation number (4) Subtracting 3 from 4, we get, 45 is equal to 2 point 5, 6 into sigma implies sigma is equal to 17 point 5, 7, 8 Substituting the value of σ in (3), we have, 25 minus mu is equal to minus 1 point 2, 8 into 3 implies mu is equal to 47 point 5. Hence, the mean is 47 point 5 kilogram and standard deviation is 17 point 5, 7, 8 kilogram.

The daily consumption of milk in a city in excess of 20,000 litres is approximately distributed as gamma variate with parameters alpha is equal to 1 divided by 10000 and beta is equal to 2. The city has a daily stock of 30,000 litres. What is the probability that the stock is insufficient on a particular day?

Solution:
If the random variable X denotes the daily consumption of milk (in litres) in a city, then the random variable Y is equal to X minus 2000 has gamma distribution with probability density function g of y is equal to 1 divided by 10 thousand square into gamma 2, into y power 2 minus 1 into e power minus Y divided by 10 thousand, where y is greater than zero.

Since the daily stock of the city is 30 thousand litres, the required probability 'p' that the stock is insufficient on a particular day is given by, p is equal to probability of X greater than 30 thousand Is equal to probability of Y greater than 10 thousand Is equal to integral from 10 thousand to infinity g of y dy Is equal to integral from 10 thousand to infinity y power 2 minus 1 into e power minus y divided by 10 thousand whole divided by 10 thousand square dy By putting z is equal to y divided by 10 thousand, we get Integral from 1 to infinity z into e power minus z dz Is equal to minus z into e power minus z, ranges from 1 to infinity plus integral from 1 to infinity e power minus z dz Is equal to e power minus 1 minus e power minus z, ranges from 1 to infinity Is equal to e power minus 1 plus e power minus 1 Is equal to 2 into e power minus 1.

Suppose that during rainy season on a tropical island the length of the shower has an exponential distribution with parameter 2, time being measured in minutes. What is the probability that a shower will last more than 3 minutes? If a shower has already lasted for 2 minutes, what is the probability that it will last for at least one more minute?

Solution:

Let X denote the length of shower in minutes. Hence, X follows exponential distribution with parameter theta, where theta is equal to 2
Hence, probability density function is given by,
f of x is equal to theta into e power minus x into theta
Is equal to 2 into e power minus 2 into x, where x is greater than zero.

(16)
Probability of [shower will last more than 3 minutes]
Is equal to Probability of [X greater than 3]
Is equal to integral from 3 to infinity f of x dx
Is equal to 2 into integral from 3 to infinity, e power minus 2 into x dx
Is equal to 2 into e power minus 2 into x divided by minus 2, ranges from 3 to infinity is equal to e power minus 6.
P[shower which has already lasted for 2 minutes, will last for at least one more minute]
Because of memory less property of exponential distribution, it is same as
Probability of [shower will last for at least one more minute]
Is equal to Probability of [x greater than 1]
Is equal to integral from 1 to infinity f of x dx
Is equal to 2 into integral from 1 to infinity, e power minus 2 into x dx
Is equal to 2 into e power minus 2 into x divided by minus 2, ranges from 1 to infinity is equal to e power minus 2.

4. Practical - 3

The moment generating function of a distribution is given by 1 divided by 1 minus 2 into t whole power 4. Identify the distribution and hence find its mean and variance.

Solution:
By uniqueness theorem of moment generating function, we can always identify the distribution of a random variable. We have given moment generating function M X of (t) is equal to 1 divided by (1 minus 2 into t) whole power 4. It is similar to moment generating function of gamma distribution, which is of the form 1 divided by 1 minus t divided by beta whole power alpha.

By comparison, it is clear that a given random variable has gamma distribution with two parameters alpha is equal to 5 and beta is equal to 1 by 2 is equal to zero point 5. Also, we know that mean of gamma distribution is alpha divided by beta is equal to 5 divided by zero point 5 is equal to 10 And variance is equal to alpha divided by beta square is equal to 5 divided by zero point 5 square is equal to 20

If X is uniformly distributed with mean 1 and variance 4/3, find Probability of (X less than zero).

Solution:
Let X follows Uniform (a, b) so that f of (x) is equal to 1 divided by (b minus a), where a less than x less than b. We have given Mean is equal to (b plus a divided by 2) is equal to 1 Implies, b plus a is equal to 2 Variance is equal to (b minus a) the whole square divided by 12 is equal to 4 divided by 3 Implies (b minus a) the whole square is equal to 16 or (b minus a) is equal to plus or minus 4.

By taking b minus a is equal to minus 4, we get a is equal to minus 1 and b is equal to 3 Or by taking b minus a is equal to 4, we get a is equal to 3 and b is equal to minus 1. But in uniform distribution, we should have, a is less than b, the solution for a and b is, a is equal to minus 1 and b is equal to 3

Therefore, f of (x) is equal to 1 divided by 4, where minus 1 less than x less than 3. Hence, we can find Probability that X less than zero is equal to integral from minus 1 to zero f of x dx Is equal to 1 divided by 4 into x, ranges from minus 1 to zero Is equal to 1 divided by 4.

5. Practical – 4 (Illustration on Pareto distribution)

Consider the following illustration on Pareto distribution.
Suppose that the income of a certain population has the Pareto distribution with shape parameter 3 and scale parameter 1 thousand. Find the proportion of the population with incomes between 2 thousand and 4 thousand

Let us solve this problem as follows.
Let X denotes the Income of a certain population. Hence, X has Pareto distribution with shape parameter (alpha) is equal to 3 and scale parameter (b) is equal to1 thousand.

Hence, we can write probability density function as follows.
f of x is equal to alpha into b power alpha divided by x power alpha plus 1 is equal to 3 into 1 thousand cube divided by x power 4, where x is greater than or equal to 1 thousand To find proportion of the population having income between 2 thousand and 4000 we need to find
Probability of income between 2 thousand and 4 thousand
Is equal to probability that 2 thousand less than X less than 4 thousand
Is equal to integral from 2 thousand to 4 thousand f of x d x
Is equal to 3 into 1 thousand cube into integral from 2 thousand to 4 thousand x power minus 4 dx
Is equal to 3 into 1 thousand cube into x power minus 3 divided by minus 3, ranges from 2 thousand to 4 thousand.
Is equal to 1 thousand cube into 2 thousand power minus 3 minus 4 thousand power minus 3.
On simplifying the above figures, we get Zero point 1, zero 9, 4.

If Y follows Normal distribution with parameters (5, and 16), then define the distribution of X is equal to e power Y. Obtain mean and variance. Also comment on skewness and kurtosis of the distribution of X.

Solution:
We know that, if Y follows Normal distribution with parameters mu and sigma square, then X is equal to e power y is called a log normal random variable. r^{th} raw moment of X is given by, Mu r dash is equal to e power r into mu plus r square into sigma square divided by 2 Is equal to e power 5 into r plus 16 into r square divided by 2.
Hence, first raw moment Mu 1 dash is equal to e power 5 plus 16 divided by 2 is equal to e power 13.

Second raw moment is given by, Mu 2 dash is equal to e power 2 into mu plus 2 into sigma square is equal to e power 2 into 5 plus 2 into 16 is equal to e power 42. Variance of the distribution is given by, Mu 2 is equal to mu 2 dash minus mu 1 dash square Is equal to e power 42 minus e power 13 square is equal to e power 26 into e power 16 minus 1 We know that coefficient of skewness Beta 1 is equal to e power 3 into sigma square, minus 3 into e power sigma square plus 2 the whole square divided by e power sigma square minus 1 whole cube Is equal to e power 3 into 16 minus 3 into e power 16 plus 2 whole square divided by e power 16 minus 1 whole cube, which is greater than zero. Coefficient of

Kurtosis is given by, Beta 2 is equal to e power 4 into sigma square plus 2 into e power 3 into sigma square plus 3 into e power 2 into sigma square minus 3 Is equal to e power 64 plus 2 into e power 48 plus 3 into e power 32 minus 3 Is equal to 6 point 2, 3, 5, 1 into 10 power 27, which is greater than 3. Hence, X is positively skewed and has leptokurtic distribution.

Our learning in this session, where we have understood:

- The definition of trinomial distribution
- The application of trinomial distribution
- The properties of trinomial distribution
- The illustrations on trinomial distribution