Subject	Statistics
Year	1st Year B.Sc
Paper no	05
Paper Name	Probability Distributions-1
Topic no	23
Topic name	Sketching Distribution Functions and Density Functions
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E-Learning Module on Sketching Distribution Functions and Density Functions

Learning Objectives

By the end of this session, you will be able to:

- Sketch the probability and distribution function of different distributions, namely:
    - Uniform distribution
    - Exponential distribution
    - Normal distribution
    - Beta distribution
    - Gamma distribution
    - Cauchy distribution
    - Laplace distribution
    - Pareto distribution

Introduction

Let us sketch the probability density function and distribution functions of the different distributions, which we have studied in paper 5 and also identify the nature of the distribution as the value of the parameter changes.

Let f(x) denote probability density function and F(x) denote distribution function.

Uniform Distribution

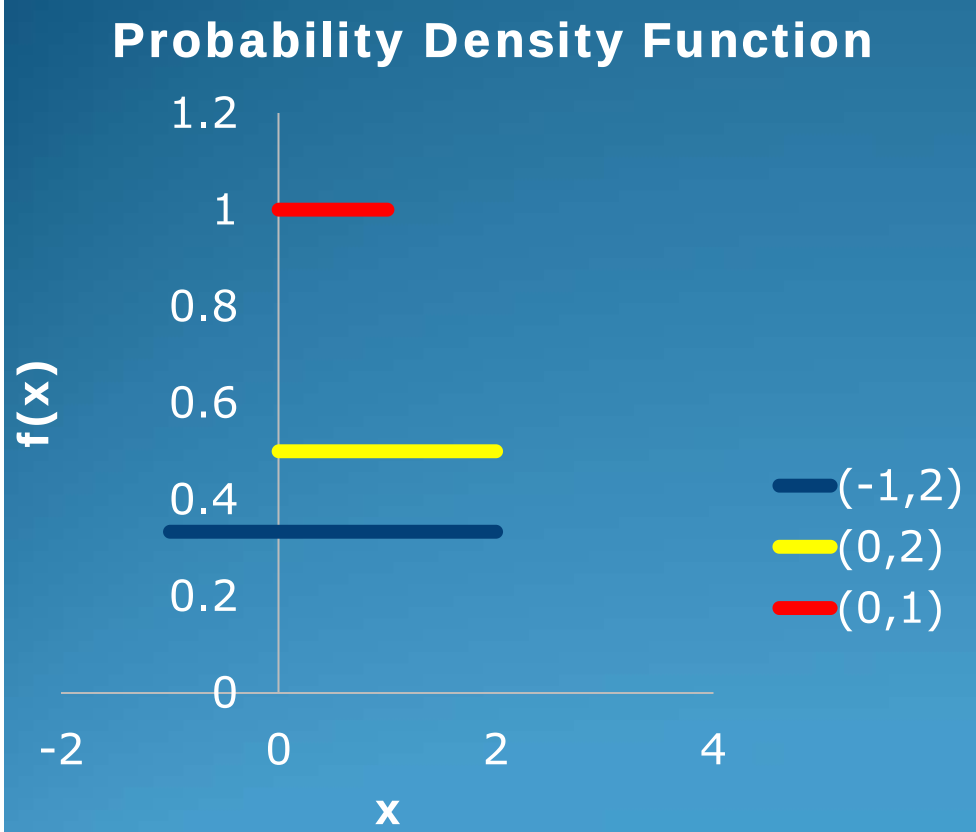
Sketch probability density function and probability distribution function of uniform distribution for different values of the parameters (a, b) and study the nature of the distribution.

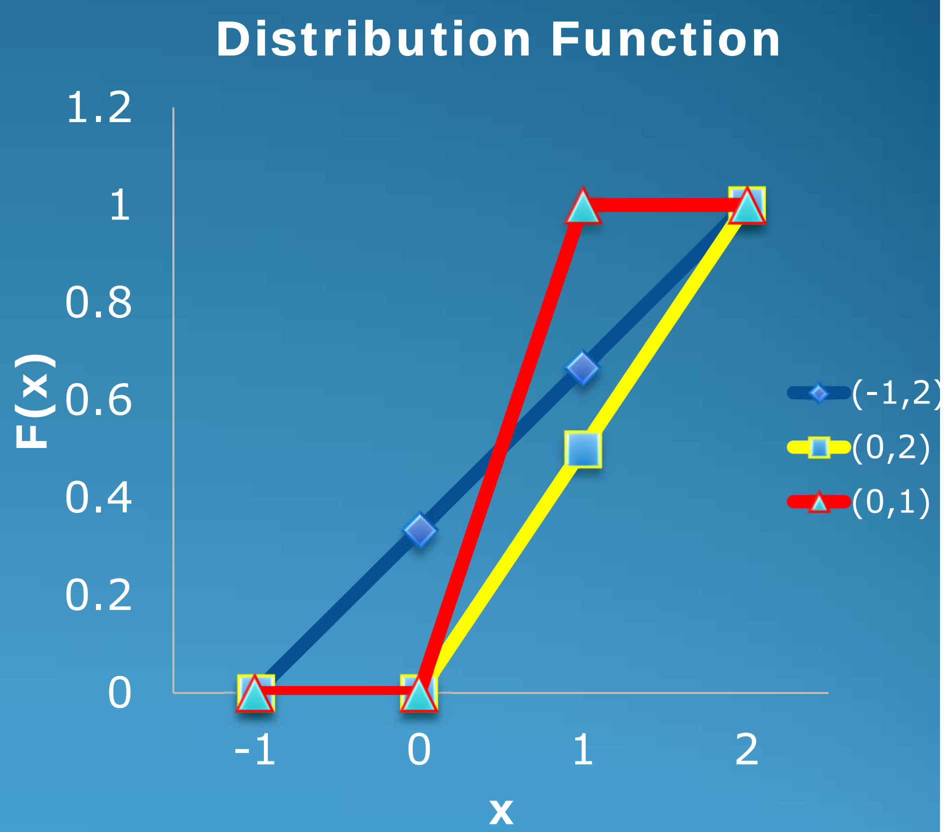
Solution:

The pdf of uniform distribution is given by,

$f(x)={}^{1}/_{[b-a]}$, $a<x<b$

$F(x)={}^{(x-a)}/_{(b-a)}$, $a<x<b$


Probability Density Function
1.2
1
0.8
0.6
0.4
0.2
0
f(x)
-2
0
2
4
x
(-1,2)
(0,2)
(0,1)

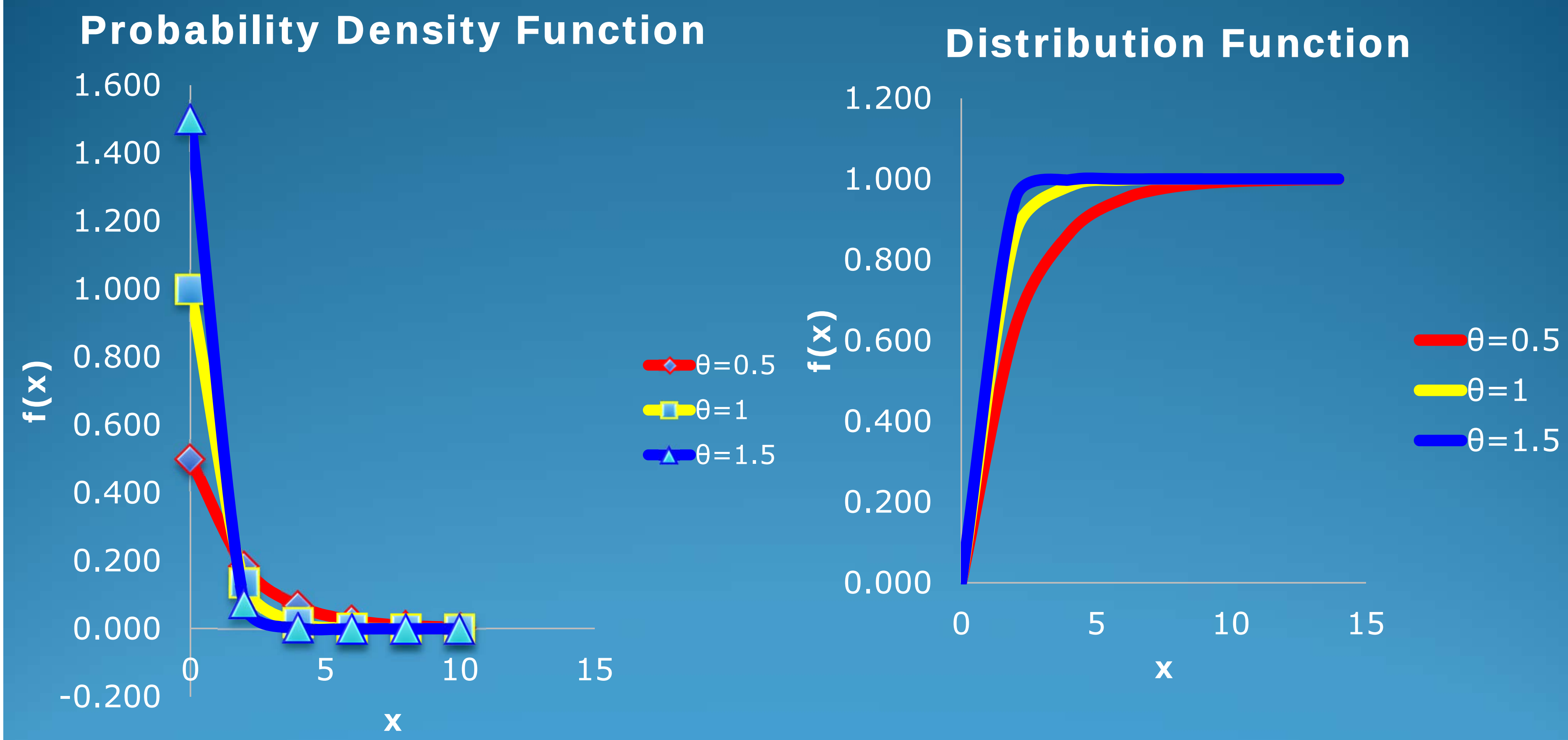

Distribution Function
1.2
1
0.8
0.6
0.4
0.2
0
F(x)
-1
0
1
2
x
(-1,2)
(0,2)
(0,1)

Exponential Distribution

Sketch probability density function and distribution function of exponential distribution for different values of the parameter θ and comment on the behaviour of the distribution.

Solution:

The pdf of exponential distribution is given by, $f(x)=\theta e^{-x\theta}$, $x>0$ and its distribution function is given by, $F(x)=1- e^{-x\theta}$


Probability Density Function
1.600
1.400
1.200
1.000
0.800
0.600
0.400
0.200
0.000
-0.200
f(x)
0
5
10
15
x
θ=0.5
θ=1
θ=1.5
Distribution Function
1.200
1.000
0.800
0.600
0.400
0.200
0.000
f(x)
0
5
10
15
x
θ=0.5
θ=1
θ=1.5

Normal Distribution

Sketch probability density function and distribution function of normal distribution for different values of the parameter μ and σ^2 and comment on the behaviour of the distribution.

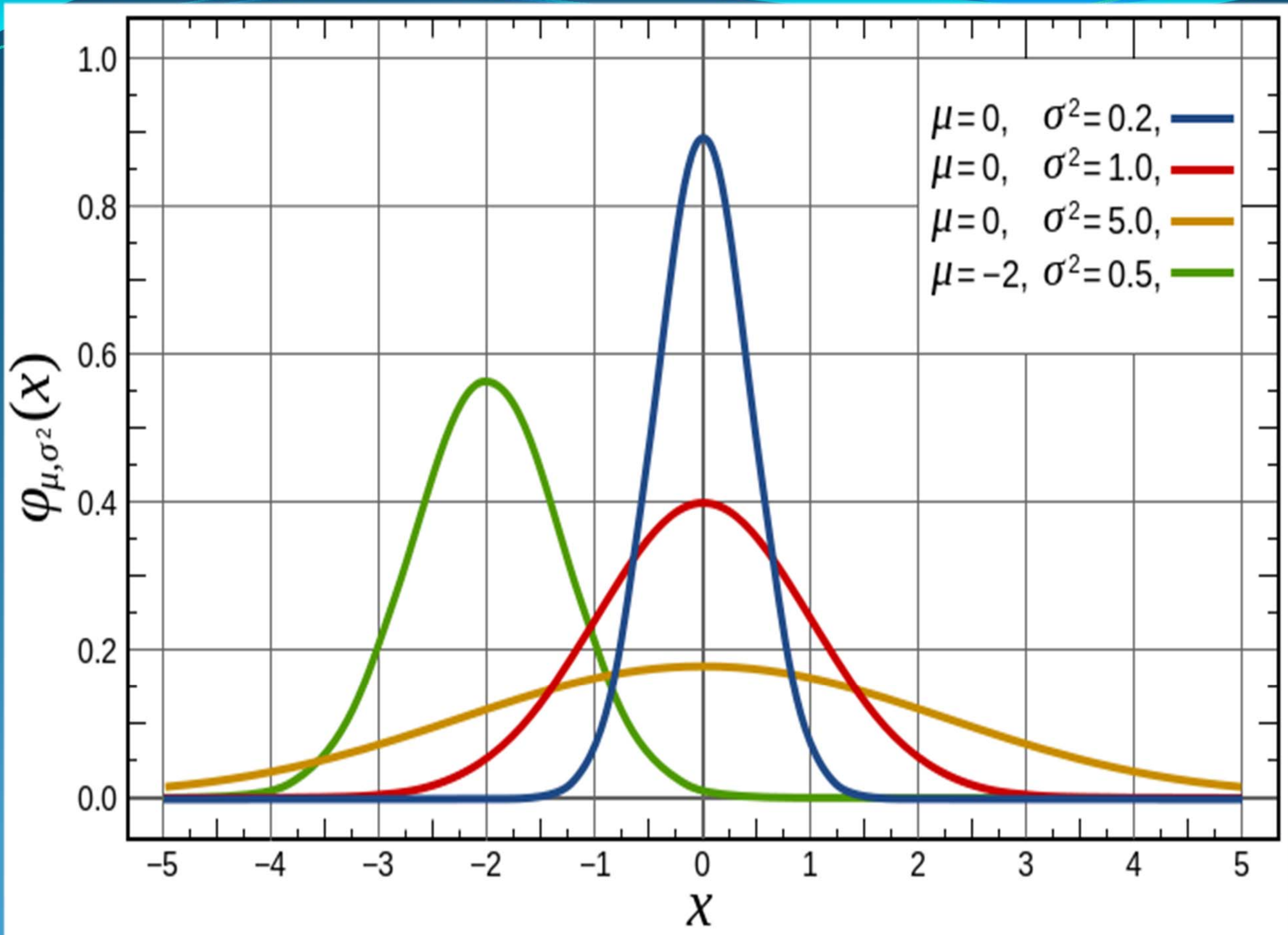
Solution:

The pdf of normal distribution is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \begin{array}{c} -\infty < x < \infty \\ ; -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

Its distribution function is given by,

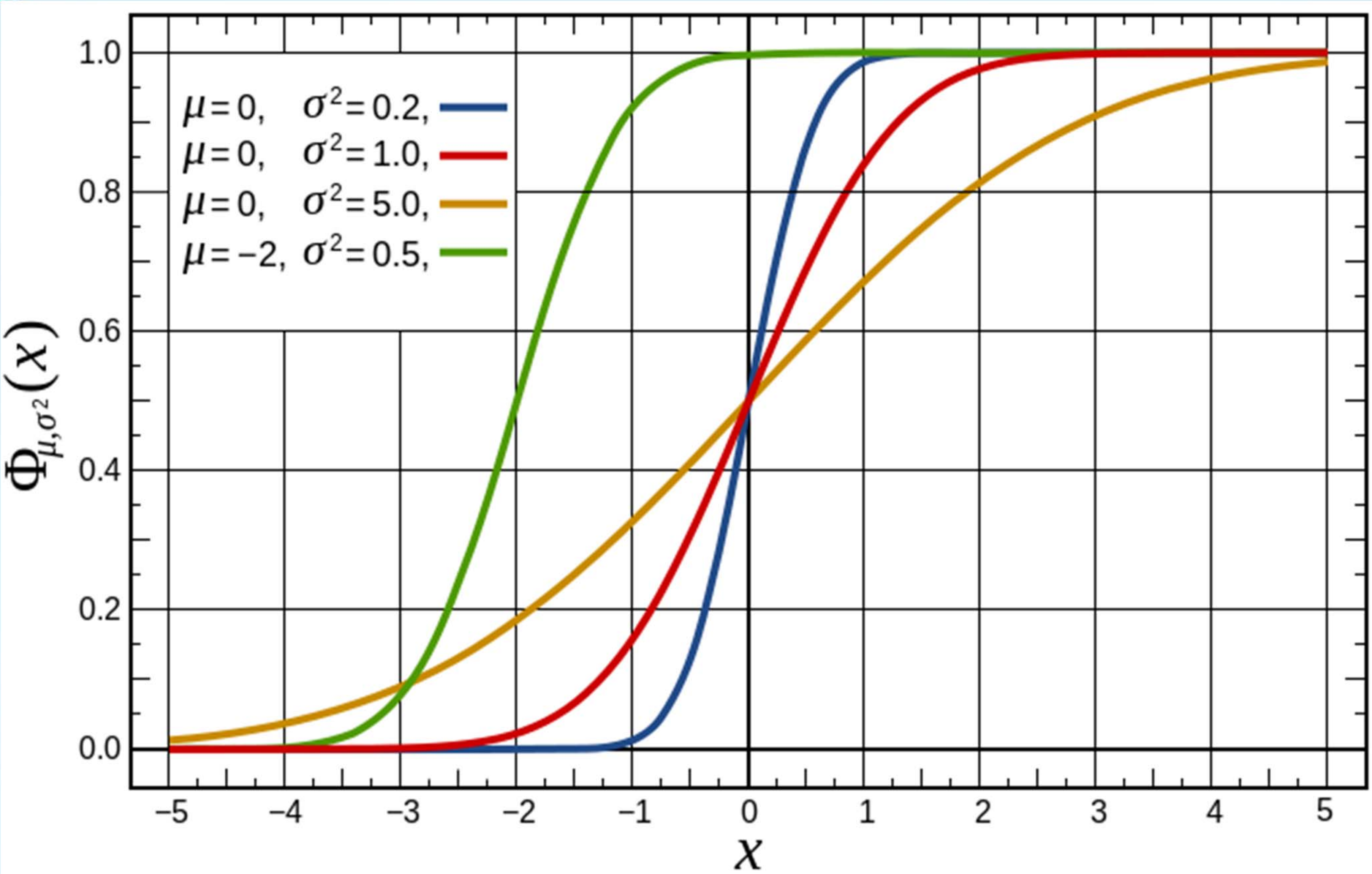
$$F(x) = \Phi_{\mu,\sigma^2}(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



$\mu=0, \quad \sigma^2=0.2,$
$\mu=0, \quad \sigma^2=1.0,$
$\mu=0, \quad \sigma^2=5.0,$
$\mu=-2, \quad \sigma^2=0.5,$
$\varphi_{\mu,\sigma^2}(x)$
x
1.0
0.8
0.6
0.4
0.2
0.0
−5
−4
−3
−2
−1
0
1
2
3
4
5

As σ^2 decreases, the values taken by the variable are in the smaller range and the curve of probability density function flattens. On the other hand, as σ^2 increases, the peakedness of the curve also increases.

As the value of the parameter μ changes, the curve of probability density function shifts accordingly with the value and is always symmetric about this value.



1.0
0.8
0.6
0.4
0.2
0.0
$\mu=0,\ \sigma^2=0.2,$
$\mu=0,\ \sigma^2=1.0,$
$\mu=0,\ \sigma^2=5.0,$
$\mu=-2,\ \sigma^2=0.5,$
$\Phi_{\mu,\sigma^2}(x)$
−5 −4 −3 −2 −1 0 1 2 3 4 5
x

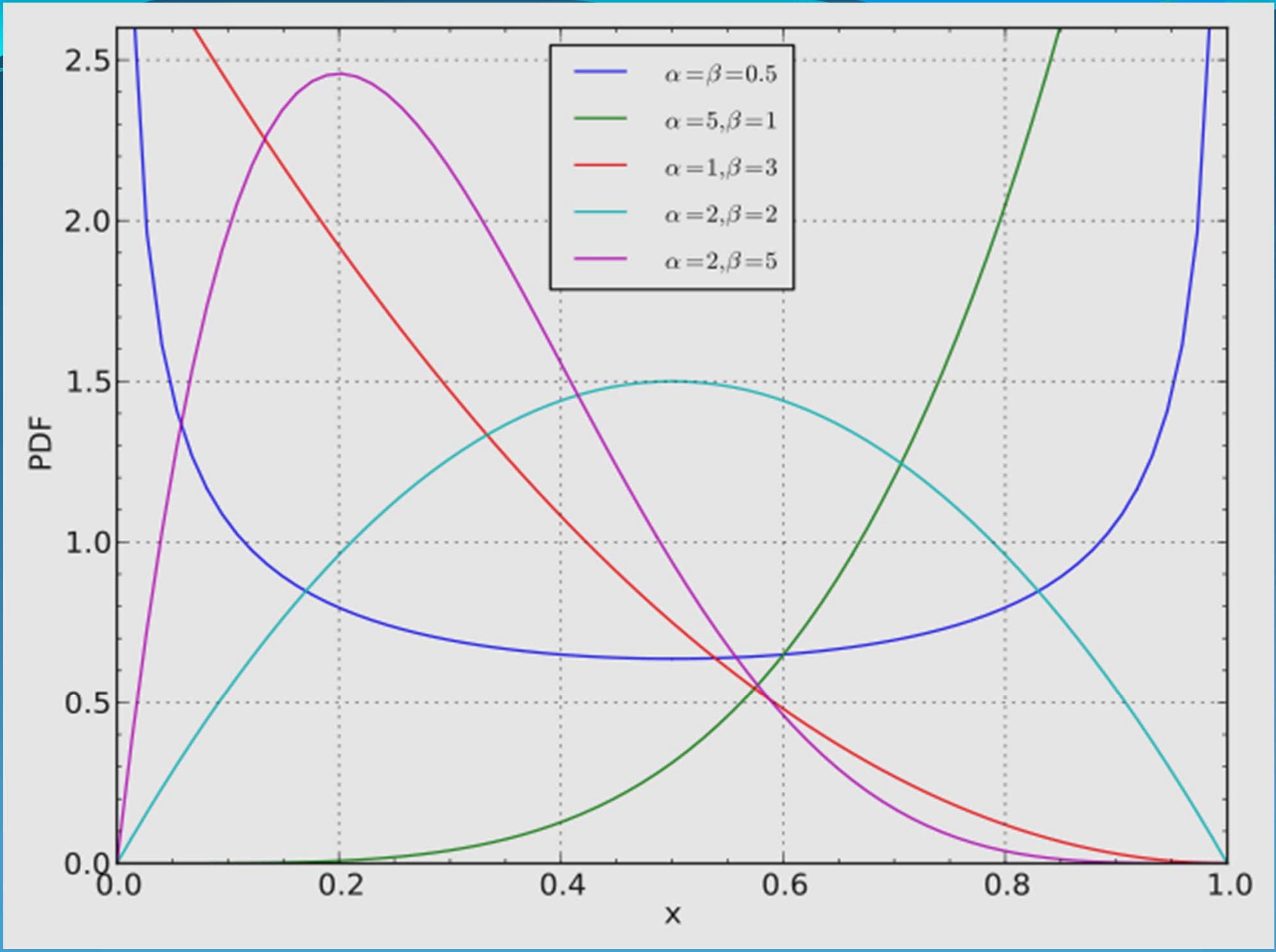
Beta Distribution

Sketch the probability density function and distribution function of the beta distribution of first kind for different values of the parameters m=α and n=β. And observe the shape of the curves.

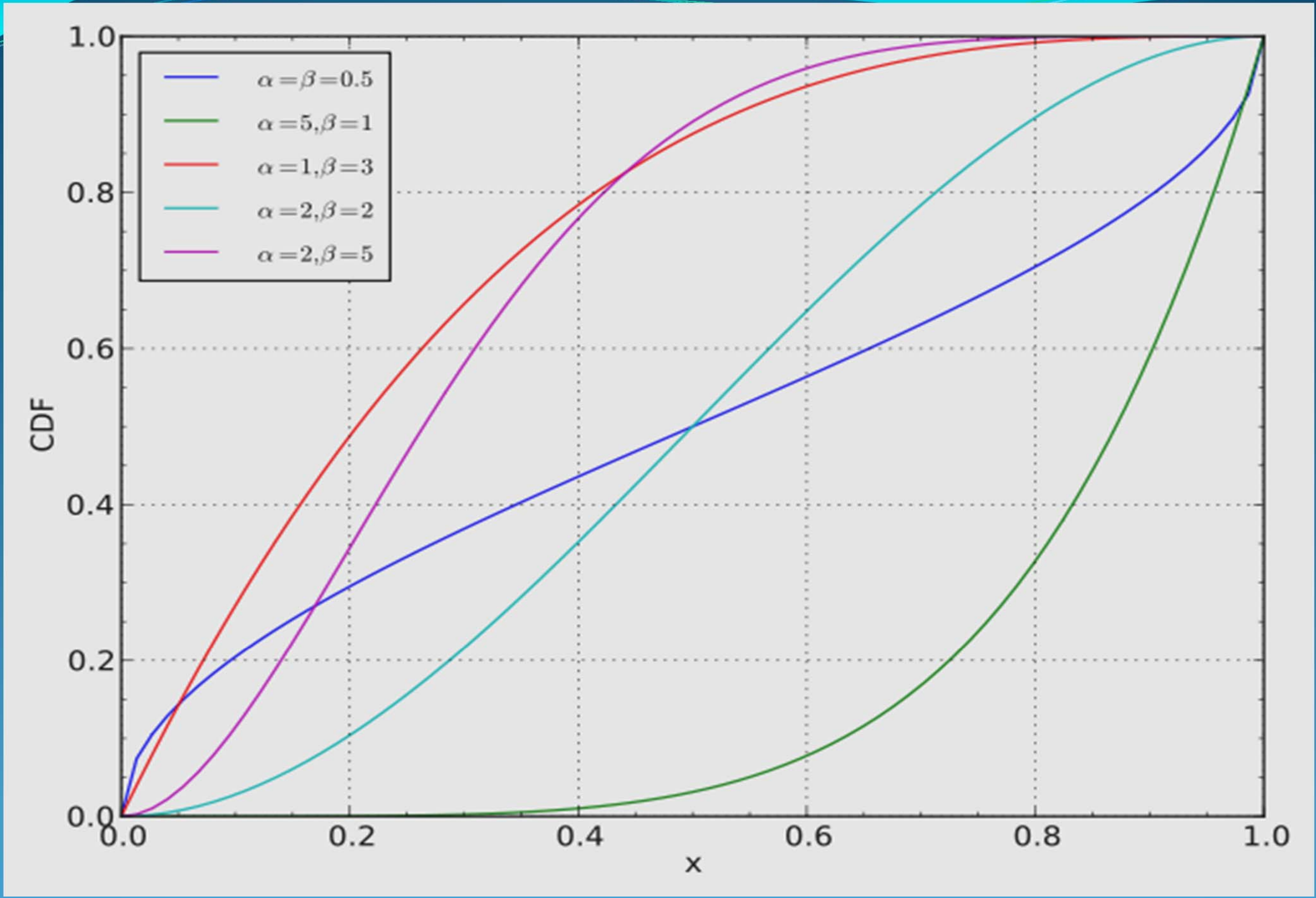
Solution: We know that for beta distribution of first kind with parameters α and β, the pdf and cumulative distribution function is given by,

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1$$

$$cdf = \frac{1}{B(\alpha,\beta)} \int_0^x x^{\alpha-1}(1-x)^{\beta-1} dx$$



$\alpha=\beta=0.5$
$\alpha=5, \beta=1$
$\alpha=1, \beta=3$
$\alpha=2, \beta=2$
$\alpha=2, \beta=5$
PDF
x
0.0
0.5
1.0
1.5
2.0
2.5
0.2
0.4
0.6
0.8
1.0



$\alpha=\beta=0.5$
$\alpha=5, \beta=1$
$\alpha=1, \beta=3$
$\alpha=2, \beta=2$
$\alpha=2, \beta=5$
CDF
x
0.0
0.2
0.4
0.6
0.8
1.0

Gamma Distribution

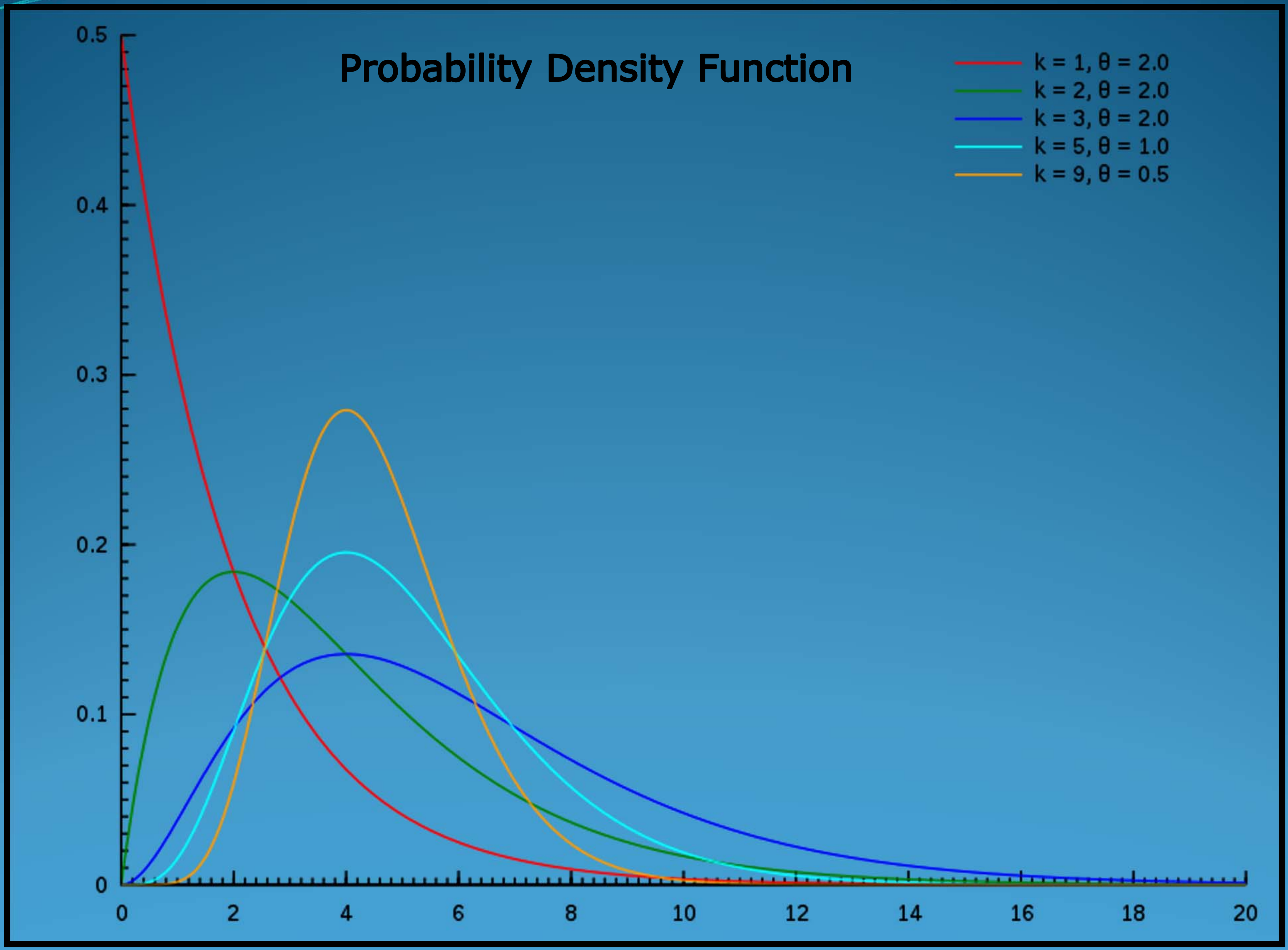
Let X~Gamma(k, θ). Sketch pdf and cdf of the distribution for different values of the parameters and comment on the behaviour of the distribution.

Solution:

The pdf and cdf are given by,

$$f(x) = \frac{\theta^k}{\Gamma k} e^{-\theta x} x^{k-1}, x > 0$$

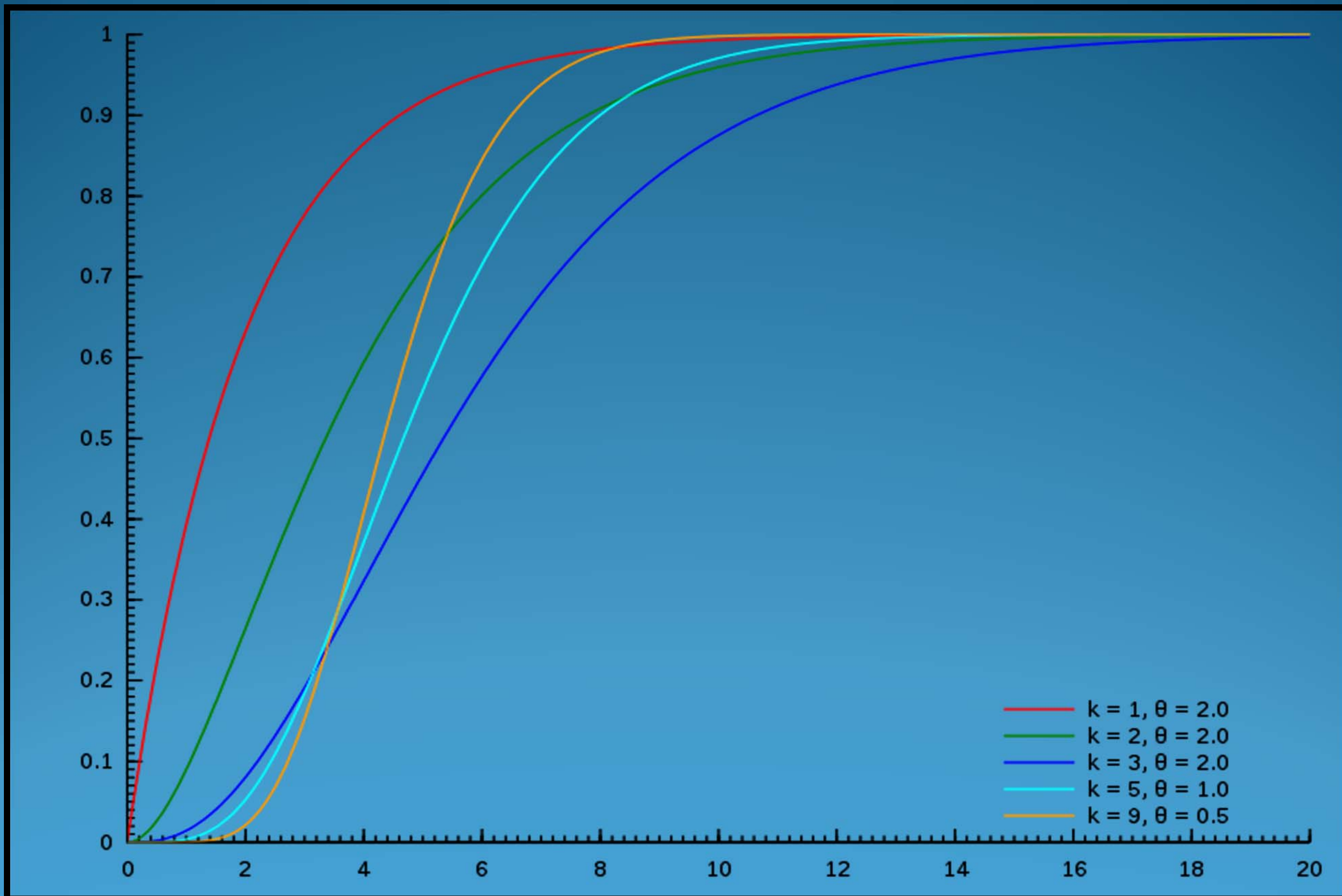
$$F(x) = \frac{\theta^k}{\Gamma k} \int_0^x e^{-\theta x} x^{k-1} dx$$


Probability Density Function
k = 1, θ = 2.0
k = 2, θ = 2.0
k = 3, θ = 2.0
k = 5, θ = 1.0
k = 9, θ = 0.5
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
18
20

- Observe that when k=1, the curve is similar to that of exponential curve. Hence, exponential distribution is a particular case of gamma distribution that is when k = 1.

- For smaller values of k, the distribution is positively skewed. As k increases, we can notice that the curve becomes more and more symmetric. Hence, for large values of k, gamma distribution tends to be a normal distribution.

Cumulative Distribution Function



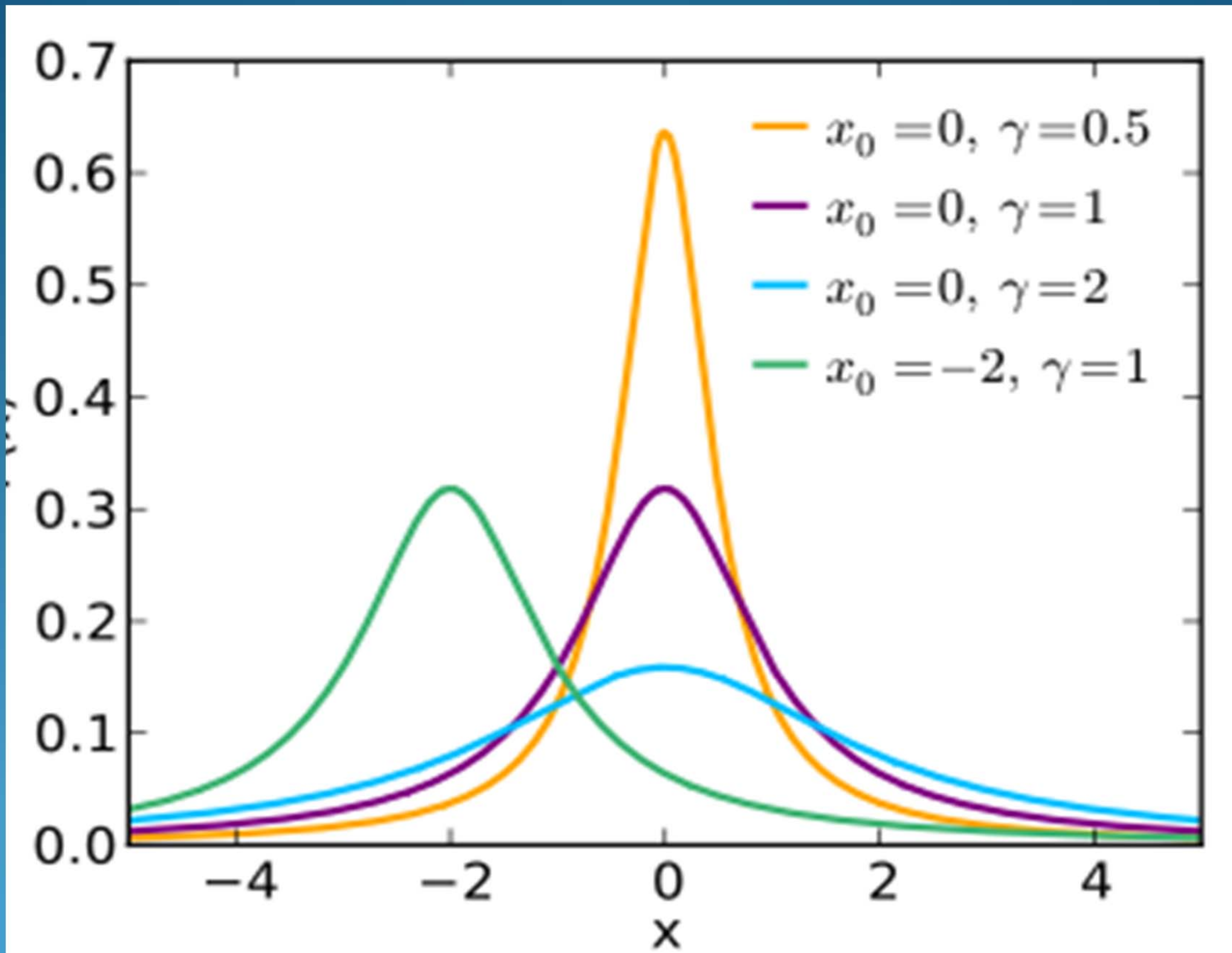
Cauchy Distribution

Let X~Cauchy(x_0,γ). Sketch the pdf and cdf of the distribution for different values of the parameters and comment on the behaviour of the distribution.

Solution: We know that, the pdf and cdf of Cauchy distribution is given by,

$$f(x) = \frac{\gamma}{\pi\left[\gamma^2 + (x - x_0)^2\right]}, -\infty < x_0 < \infty; \gamma > 0$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{\gamma}{\left[\gamma^2 + (x - x_0)^2\right]} dx$$

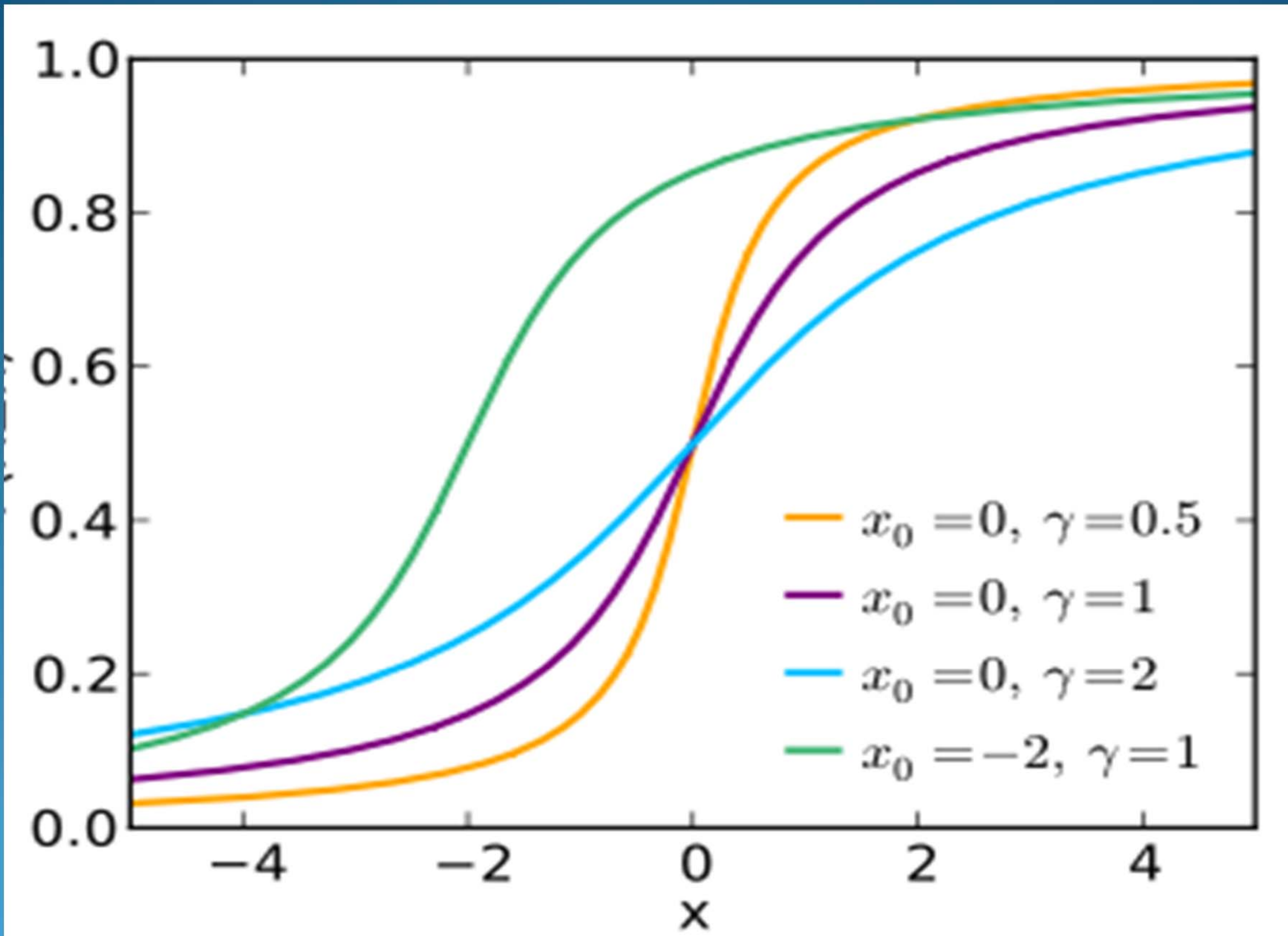


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−4
−2
0
2
4
x
$x_0 = 0,\ \gamma = 0.5$
$x_0 = 0,\ \gamma = 1$
$x_0 = 0,\ \gamma = 2$
$x_0 = -2,\ \gamma = 1$

Observe that the curve of Cauchy distribution is bell shaped and is symmetric about the parametric value x_0.

As the value of gamma increases, the curve becomes flatter than the smaller values of gamma.

As the value of the parameter x_0 changes, the curve shifts its position accordingly.



1.0
0.8
0.6
0.4
0.2
0.0
−4
−2
0
2
4
x
$x_0 = 0,\ \gamma = 0.5$
$x_0 = 0,\ \gamma = 1$
$x_0 = 0,\ \gamma = 2$
$x_0 = -2,\ \gamma = 1$

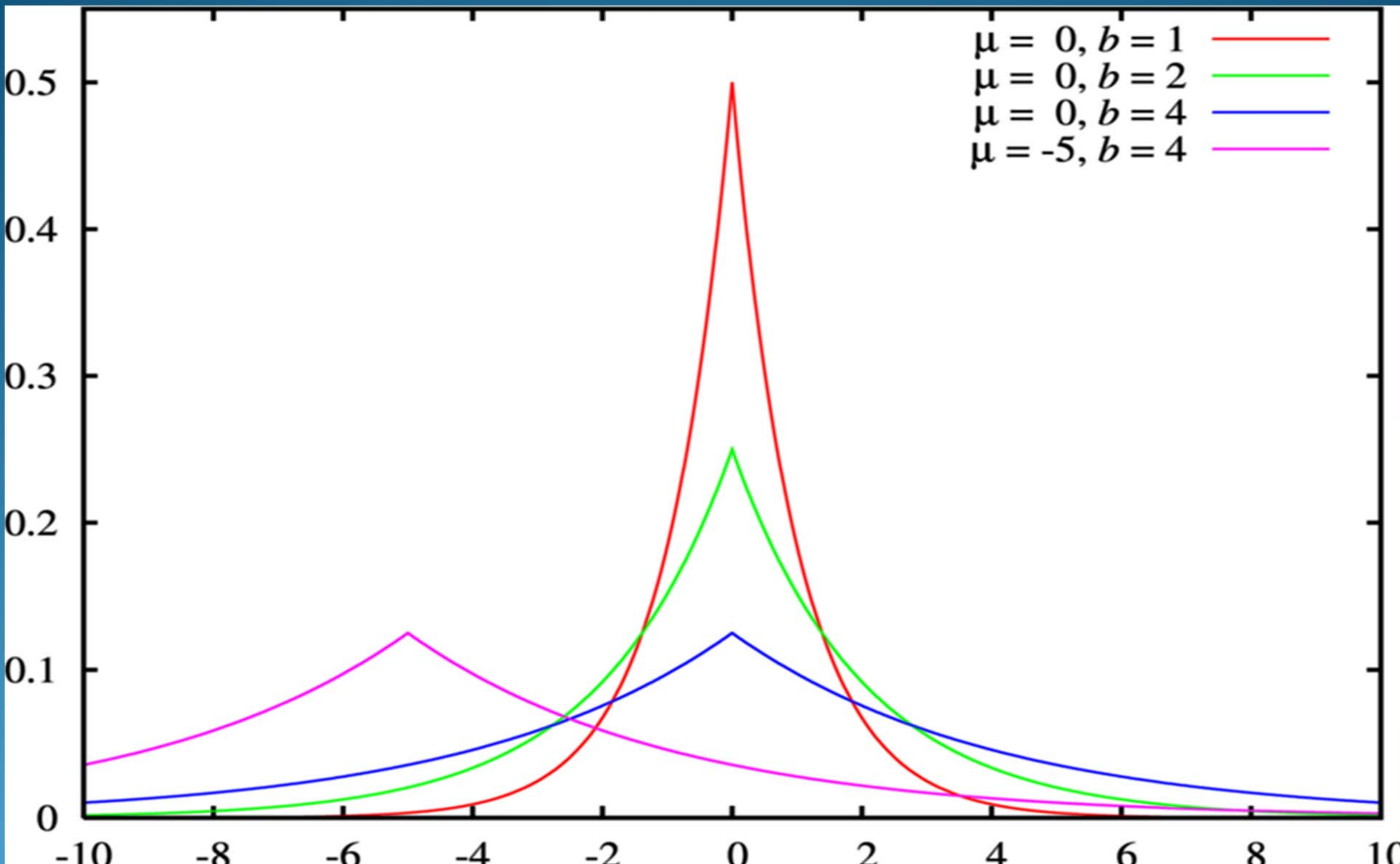
Laplace Distribution

Let X~Laplace(μ, b). Sketch pdf and cdf of the distribution for different values of the parameters and comment on the behaviour of the distribution.

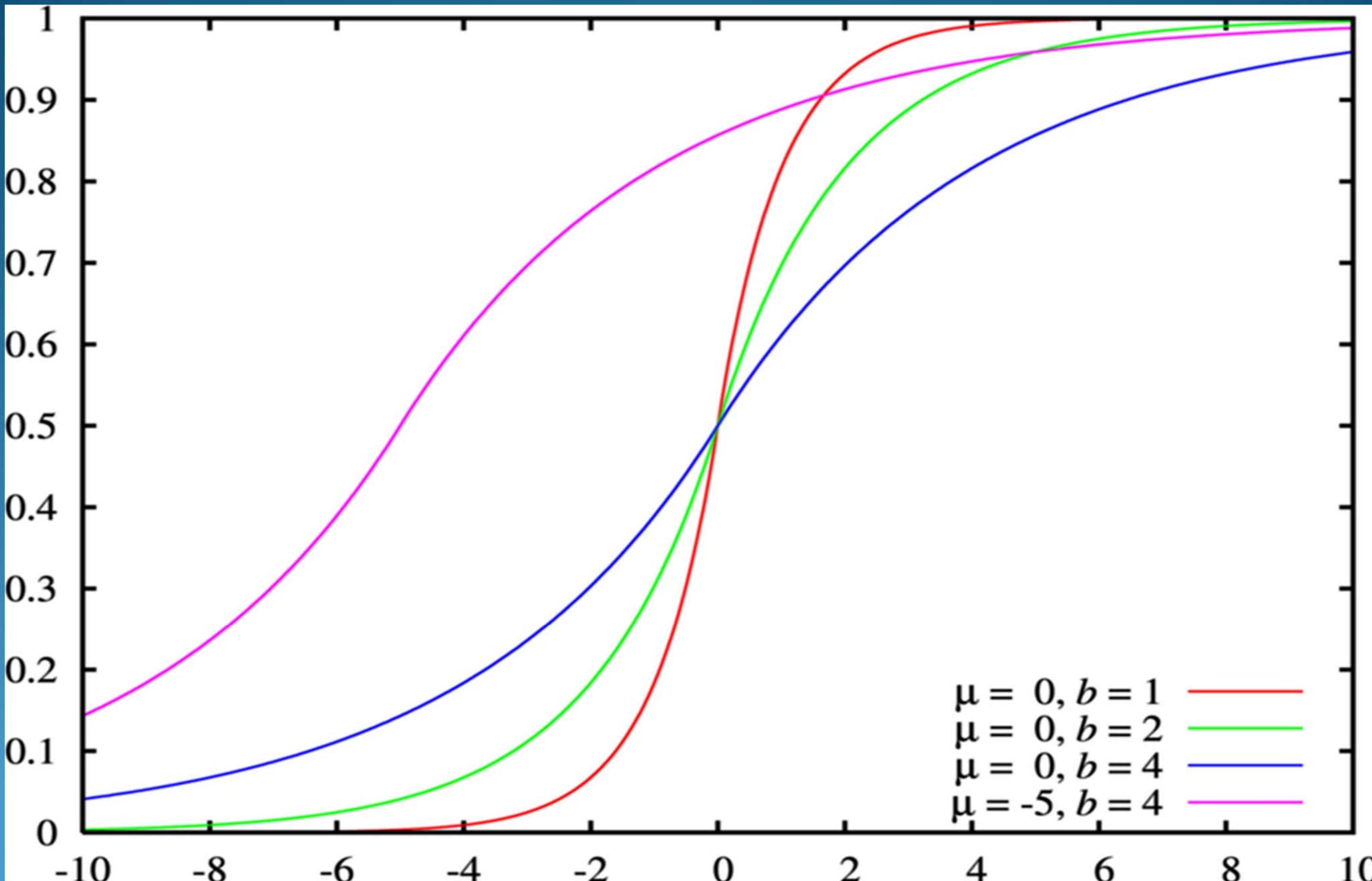
Solution: We know that, the pdf and cdf of Laplace distribution is given by,

$$f(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}, \quad -\infty < x < \infty, \quad b > 0$$

$$F(x) = \frac{1}{2b} \int_{-\infty}^{x} e^{-\frac{|x-\mu|}{b}} dx$$



μ = 0, b = 1
μ = 0, b = 2
μ = 0, b = 4
μ = -5, b = 4
0.5
0.4
0.3
0.2
0.1
0
-10
-8
-6
-4
-2
0
2
4
6
8
10



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-8
-6
-4
-2
0
2
4
6
8
10
μ = 0, b = 1
μ = 0, b = 2
μ = 0, b = 4
μ = -5, b = 4

Pareto Distribution

Sketch pdf and cdf of Laplace distribution for different values of the parameters α and b and comment on the behaviour of the distribution.

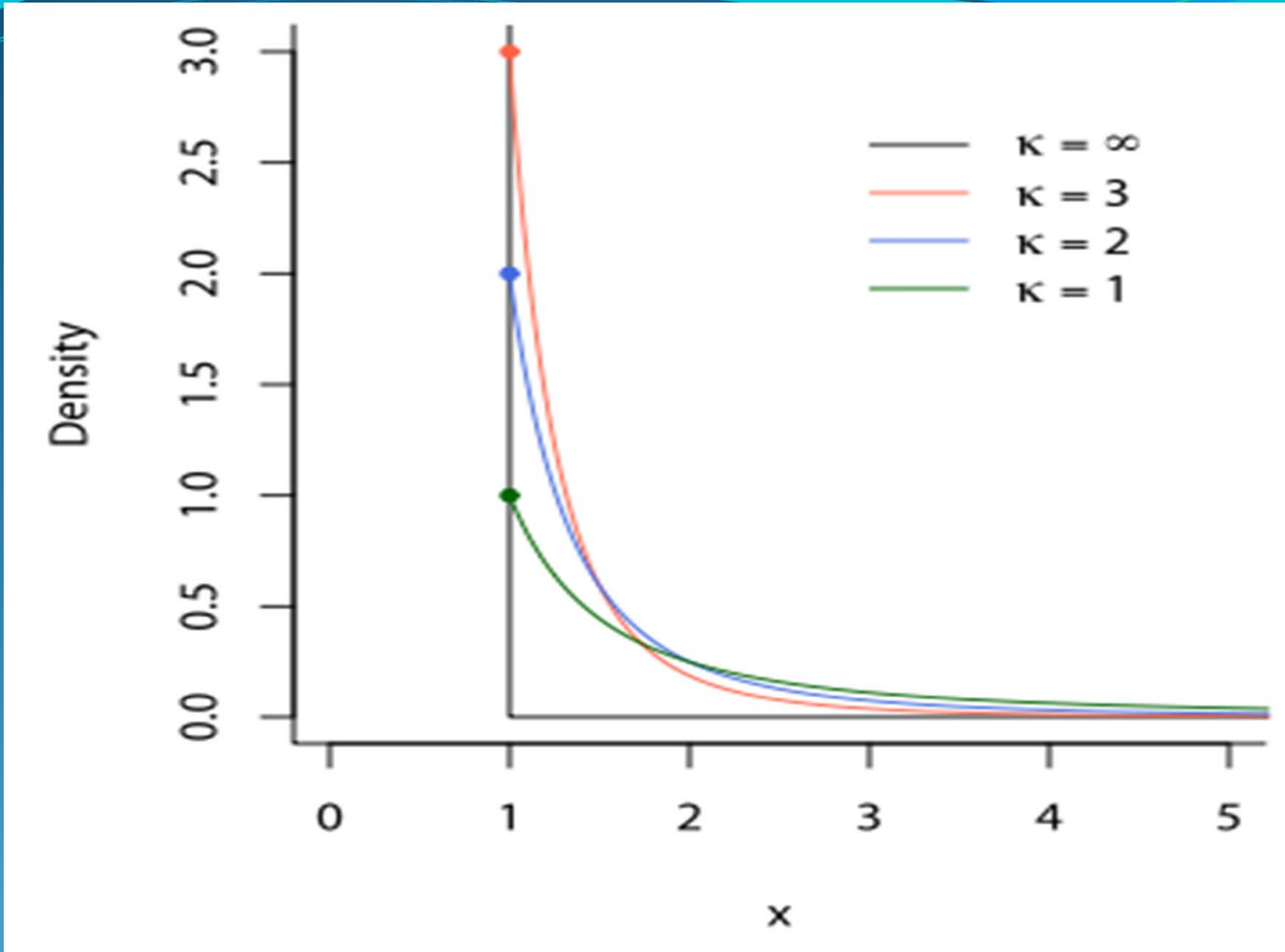
Solution:

We know that pdf of the distribution is

$$f(x) = \frac{\alpha b^{\alpha}}{x^{\alpha+1}}; x \geq b, \alpha > 0$$

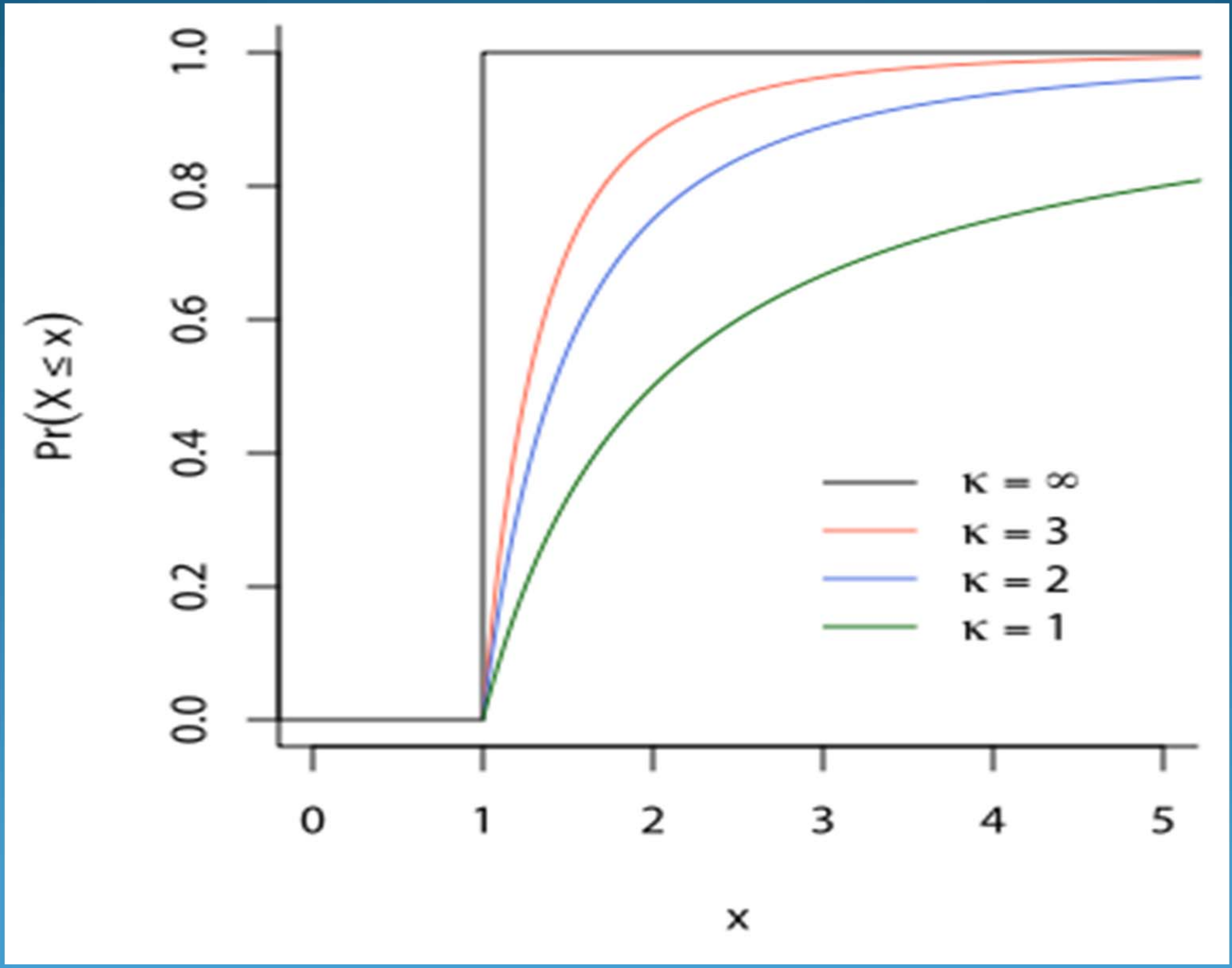
Cumulative distribution function is given by,

$$F(x) = 1 - \frac{b^{\alpha}}{x^{\alpha}}$$


κ = ∞
κ = 3
κ = 2
κ = 1
Density
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
x

Observe that probability density function of Pareto distribution is strictly a decreasing function.

Since we have taken b=1, all the curves start from 1 of the x-axis. As the value of k increases, the curve becomes steeper.



1.0
0.8
0.6
0.4
0.2
0.0
Pr(X ≤ x)
0
1
2
3
4
5
x
κ = ∞
κ = 3
κ = 2
κ = 1