E-Learning Module on Drawing random samples from Bivariate Normal Distribution

Learning Objectives

At the end of this session, you will be able to know:

- ❖ Method of drawing random samples from bivariate normal distribution
- ❖ Illustration on drawing samples with different possible values of parameter.
- ❖ When rho is equal to zero, bivariate samples will be two independent samples from two normal distributions

Introduction

A bivariate random variable (X,Y) is said to follow Bivariate normal distribution with parameters

μ_1, the of mean X,

μ_2, the mean of Y,

σ_1^2, the variance of X

σ_2^2, the variance of Y and

ρ, the coefficient of correlation between X and Y if its pdf is given by,

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)}$$

Range of the distribution is as below

$-\infty<(x,y)<\infty$, $-\infty<(\mu_1, \mu_2)<\infty$, $(\sigma_1, \sigma_2)>0$ $-1<\rho<1$

Generating Sample

The following algorithm can be used to generate sample from the bivariate normal distribution:

Let z_1 and z_2 be independent draws from the standard normal distribution, $N(0,1)$.

Then x and y, calculated as follows will have a joint bivariate normal distribution with parameters $(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$:

$x=\mu_1+\sigma_1 z_1$

$y=\mu_2+\sigma_2[z_1\rho+z_2 \sqrt{(1-\rho^2)}]$

Exercise 1

Draw 10 random samples from Bivariate normal distribution with parameters (μ_1=30, μ_2=47, σ_1^2=49, σ_2^2=81, ρ=0.75) using following 20 random numbers. 268, 166, 838, 280, 455, 465, 696, 020, 469, 797, 346, 974, 418, 615, 159, 659, 157, 628, 465, 705.

Solution:

Since we need to draw 10 bivariate random samples, let us consider 20 random samples. These 20 random samples are taken pair wise to get bivariate sample.

Since we have 3 digit random numbers, first we convert random numbers into probability by dividing them by 1000.

Now let us find corresponding points to these probabilities, which are named as z_1 and z_2. then using the relation, $x=\mu_1+\sigma_1 z_1=30+7z_1$ *and* $y=\mu_2+\sigma_2[z_1\rho+z_2\sqrt{(1-\rho^2)}]$

$$=47+9((z_1*0.75)+(z_2\sqrt{(1-0.75^2)}))$$

, we obtain the random samples.

The calculations are done in the following table.

Random Numbers		$P(Z<z_1)$	$P(Z<z_2)$	z_1	z_2	X	Y
268	166	0.268	0.166	-0.62	-0.97	25.66	37.04
838	280	0.838	0.280	0.99	-0.58	36.93	50.23
455	465	0.455	0.465	-0.11	-0.09	29.23	45.72
696	020	0.696	0.020	0.51	-2.05	33.57	38.24
469	797	0.469	0.797	-0.09	0.83	29.37	51.33
346	974	0.346	0.974	-0.40	2.00	27.20	56.21
418	615	0.418	0.615	-0.21	0.29	28.53	47.31
159	656	0.159	0.656	-1.00	0.40	23.00	42.63
157	628	0.157	0.628	-1.01	0.33	22.93	42.15
465	705	0.465	0.705	-0.09	0.54	29.37	49.61

Exercise 2

Draw 10 random samples from Bivariate normal distribution with parameters (μ_1=0 μ_2=0, σ_1=1 σ_2=1 and ρ=0.5) using following 20 random numbers. 2682, 1663, 8389, 2805, 4557, 4658, 6960, 0020, 4695, 7971, 3468, 9744, 4185, 6154, 1592, 6599, 1573, 6288, 4650, 7053.

Solution:

Since we need to draw 10 bivariate random samples, let us consider 20 random samples. These 20 random samples are taken pair wise to get bivariate sample.

Since we have 4 digit random numbers, first we convert random numbers into probability by dividing them by 10000.

Now let us find corresponding points to these probabilities, which are named as z_1 and z_2. then using the relation, $x=\mu_1+\sigma_1 z_1=z_1$ *and* $y=\mu_2+\sigma_2[z_1\rho+z_2\sqrt{(1-\rho^2)}]$

$=(z_1*0.5)+(z_2\sqrt{(1-0.5^2)})$

, we obtain the random samples.

The calculations are done in the following table.

Random Numbers		$P(Z<z_1)$	$P(Z<z_2)$	z_1	z_2	x	Y
2682	1663	0.2682	0.1663	-0.62	-0.97	-0.62	-0.78
8389	2805	0.8389	0.2805	0.99	-0.58	0.99	1.24
4557	4658	0.4557	0.4658	-0.11	-0.09	-0.11	-0.14
6960	0020	0.6960	0.0020	0.51	-2.05	0.51	0.64
4695	7971	0.4695	0.7971	-0.08	0.83	-0.08	-0.10
3468	9744	0.3468	0.9744	-0.40	2.00	-0.40	-0.50
4185	6154	0.4185	0.6154	-0.21	0.29	-0.21	-0.26
1592	6569	0.1592	0.6569	-1.00	0.40	-1.00	-1.25
1573	6288	0.1573	0.6288	-1.01	0.33	-1.01	-1.26
4650	7053	0.4650	0.7053	-0.09	0.54	-0.09	-0.11

Exercise 3

Draw 10 random samples from Bivariate normal distribution with parameters (μ_1=30, μ_2=47, σ_1^2=49, σ_2^2=81, ρ=0) using following 20 random numbers. 268, 166, 838, 280, 455, 465, 696, 020, 469, 797, 346, 974, 418, 615, 159, 659, 157, 628, 465, 705.

Solution:

Since we need to draw 10 bivariate random samples, let us consider 20 random samples. These 20 random samples are taken pair wise to get bivariate sample.

Since we have 3 digit random numbers, first we convert random numbers into probability by dividing them by 1000.

Now let us find corresponding points to these probabilities, which are named as z_1 and z_2. then using the relation, $x=\mu_1+\sigma_1 z_1=30+7z_1$ *and* $y=\mu_2+\sigma_2[z_1\rho+z_2\sqrt{(1-\rho^2)}]$

$=47+9((z_1*0)+(z_2\sqrt{(1-0^2)}))= 47+9z_2$

, we obtain the random samples.

The calculations are done in the following table.

Random Numbers		$P(Z<z_1)$	$P(Z<z_2)$	z_1	z_2	x	Y
268	166	0.268	0.166	-0.62	-0.97	25.66	38.27
838	280	0.838	0.280	0.99	-0.58	36.93	41.78
455	465	0.455	0.465	-0.11	-0.09	29.23	46.19
696	020	0.696	0.020	0.51	-2.05	33.57	28.55
469	797	0.469	0.797	-0.09	0.83	29.37	54.47
346	974	0.346	0.974	-0.40	2.00	27.20	65.00
418	615	0.418	0.615	-0.21	0.29	28.53	49.61
159	656	0.159	0.656	-1.00	0.40	23.00	50.60
157	628	0.157	0.628	-1.01	0.33	22.93	49.97
465	705	0.465	0.705	-0.09	0.54	29.37	51.86