Statistics

Computing Marginal and Conditional Expectations

1. Introduction

Welcome to the series of E-learning modules on computing marginal and conditional expectations.

By the end of this session, you will be able to:

- Find the marginal mean, variance and covariance.
- Find the conditional expectation and conditional variance of discrete and continuous random variables.

Given a bivariate data on (X,Y), the marginal expectations are Expectation of (X) and Expectation of (Y) and corresponding marginal Variances are Variance of (X) and Variance of (Y) and also we can find covariance, Covariance of (X, Y) .

For discrete variable,
Expectation of (X) is equal to summation x into p of (x), where p of (x) is marginal distribution of X.
Expectation of (Y) is equal to summation y into p of (y), where p of (y) is marginal distribution of Y.

For continuous bivariate random variable,
Expectation of x is equal to integral over x, x into f of x dx and Expectation of Y is equal to integral over y, y into f of y dy, where f of (x) and f of (y) are marginal probability density functions.

Similarly,
Variance of (x) is equal to Expectation of (X) square minus Expectation of X the whole square and Variance of (Y) is equal to Expectation of Y square minus Expectation of Y the whole square.
Covariance of (X, Y) is equal to Expectation of X into Y minus Expectation of X into Expectation of Y.
Similarly, we can find Conditional Expectations and Variances.

Conditional expectation of discrete random variable X given Y is equal to y_j is given by, Expectation of X given Y is equal to yj is equal to summation over i is equal to 1 to infinity, xi into Probability of X is equal to xi given Y is equal to yj

The conditional variance of X given Y is equal to y_j is likewise given by Variance of X given Y is equal to yj is equal to Expectation of X minus Expectation of X given Y is equal to yj square given Y is equal to yj.

Similarly, we can write conditional mean and conditional variance of Y given X is equal to x_i as follows:

Expectation of Y given X is equal to xi is equal to summation over j is equal to 1 to infinity yj into Probability of Y is equal to yj given X is equal to xi and Variance of Y given X is equal to xi is equal to Expectation of Y minus Expectation of Y given X is equal to xi square given X is equal to xi. Replacing summation by integral, we can obtain conditional mean and variance of continuous bivariate random variables.

2. Exercise 1

Two random variables X and Y have the following joint probability density function.
f of (x,y) is equal to k into (4 minus x minus y); where zero less than or equal to x less than or equal to 2, and zero less than or equal to y less than or equal to 2. Find constant k, Expectation of (X), Expectation of (Y), Variance of (X), Variance of (Y) and Covariance of (X, Y). Also, write Expectation of X plus Y and Expectation of X minus Y.

We can solve the above problem as follows.
Since f of x, y is a probability density function, double integral, f of x, y dx dy is equal to 1. That is, k into double integral from zero to 2, 4 minus x minus y dx dy is equal to 1 Integrating over x and simplifying we get, K into integral from zero to 2 6 minus 2 into y dy is equal to 1 Integrating over x and simplifying we get, K into 8 is equal to 1 implies, k is equal to 1 divided by 8.

Before we find Expectation and variance, we find marginal distribution of X and Y. Marginal distribution of X is given by, f of x is equal to 1 divided by 8 into integral from zero to 2 4 minus x minus y dy On integrating the function and simplifying we get, 1 divided by 8 into 6 minus 2 x By taking 2 common outside the bracket and cancelling it with the denominator we get, 1 divided by 4 into 3 minus x, where zero less than or equal to x less than or equal to 2. Marginal distribution of Y is given by, f of y is equal to 1 divided by 8 into integral from zero to 2 4 minus x minus y dx On integrating the function and simplifying we get, 1 divided by 8 into 6 minus 2 y By taking 2 common outside the bracket and cancelling it with the denominator we get, 1 divided by 4 into 3 minus y, where zero less than or equal to y less than or equal to 2.

Now, let us obtain Expectation.
Expectation of X is equal to integral from zero to 2 x into f of x dx Is equal to 1 divided by 4 into integral from zero to 2, x into 3 minus x dx On integration and simplifying we get, 5 divided by 6.
Expectation of Y is equal to integral from zero to 2 y into f of y dy Is equal to 1 divided by 4 into integral from zero to 2, y into 3 minus y dy On integration and simplifying we get, 5 divided by 6.

To find variance, first we find expectation of X square and expectation of Y square
Expectation of X square is equal to integral from zero to 2 x square into f of x dx Is equal to 1 divided by 4 into integral from zero to 2, x square into 3 minus x dx Is equal to 1

Similarly, expectation of Y square is equal integral from zero to 2, y square into f of y dy Is equal to 1 divided by 4 into integral from zero to 2 y square into 3 minus y dy is equal to 1

Variances are given as follows:
Variance of (X) is equal to Expectation of (X square) minus {Expectation of (X)} the whole square is equal to 1 minus (5 divided by 6) whole square is equal to 11 divided by 36. Variance of (Y) is equal to Expectation of (Y square) minus {Expectation of (Y)} the whole square is equal to 1 minus (5 divided by 6) whole square is equal to 11 divided by 36. We know that Covariance of (X,Y) is equal to Expectation of (X into Y) minus Expectation of (X) into Expectation of (Y) We have already found Expectation of (X) and Expectation of (Y). Now, let us obtain Expectation of (X into Y).

Expectation of X into Y is equal to double integral from zero to 2, x into y into f of x, y dx dy.
Is equal to 1 divided by 8 into integral from zero to 2 y into integral from zero to 2 x into 4 minus x minus y dx dy

By integrating the terms in the bracket and simplifying we get,
1 divided by 8 into integral from zero to 2 y into 16 divided by 3 minus 2 into y dy On integrating the function with respect to y and then simplifying we get, 2 divided by 3. Hence, covariance of X Y is equal to 2 divided by 3 minus 5 divided by 6 into 5 divided by 6 On simplifying the equation, we get, minus 1 divided by 36. We know that for any 2 variables, Expectation of X plus Y is equal to Expectation of X plus expectation of Y Is equal to 5 divided by 6 plus 5 divided by 6 is equal to 10 divided by 6 Also, expectation of X minus Y is equal to expectation of X minus Expectation of Y Is equal to 5 divided by 6 minus 5 divided by 6 is equal to zero.

3. Exercise 2

Consider an illustration on discrete data.
X and Y have a bivariate distribution given by Probability of X is equal to x intersection Y is equal to y is equal to x plus 3 into y whole divided by 24; where x y take values, 1, 1; 1, 2; 2, 1 and 2, 2

Find Expectation of (X), Expectation of (Y), Variance of (X), Variance of (Y) and Covariance of (X,Y)

We can solve the above problem as follows.
First, let us obtain marginal distribution of X and Y.Marginal distribution of X is given by Probability of X is equal to x is equal to summation over y is equal to 1 to 2 x plus 3 into y whole divided by 24 Is equal to x plus 3 into 1 whole divided by 24 plus x plus 3 into 2 whole divided by 24 Is equal to 2 into x plus 9 whole divided by 24, where x is equal to 1 and 2.Marginal distribution of Y is given by, Probability of Y is equal to y is equal to summation over x is equal to 1 to 2, x plus 3 into y whole divided by 24 Is equal to 1 plus 3 into y whole divided by 24 plus 2 plus 3 into y whole divided by 24 Is equal to 3 plus 6y divided by 24 where y is equal to 1 and 2.

Now, let us find the expectations.
Expectation of X is equal to summation over x is equal to 1 to 2 x into Probability of X is equal to x

Is equal to summation over x is equal to 1 to 2 into x into 2 into x plus 9 divided by 24.
Is equal to 1 into 2 into 1 plus 9 divided by 24 plus 2 into 2 into 2 plus 9 divided by 24
Is equal to 1 into 11 by 24 plus 2 into 13 by 24 is equal to 37 by 24.
Expectation of Y is equal to summation over y is equal to 1 to 2, y into probability of Y is equal to y
Is equal to summation over y is equal to 1 to 2, y into 3 plus 6 into y whole divided by 24.
Is equal to 1 into 3 plus 6 into 1 whole divided by 24 plus 2 into 3 plus 6 into 2 whole divided by 24.
Is equal to 1 into 9 by 24 plus 2 into 15 by 24 is equal to 39 by 24 is equal to 13 by 8.

Before we find the variance, let us find expectation of X square and Expectation of Y square.
Expectation of X square is equal to summation over x is equal to 1 to 2, x square into Probability of X is equal to x
Is equal to summation over x is equal to 1 to 2 x square into 2 into x plus 9 divided by 24
Is equal to 1 square into 2 into 1 plus 9 divided by 24 plus 2 square into 2 into 2 plus 9 divided by 24
Is equal to 1 into 11 by 24 plus 4 into 13 by 24 is equal to 63 by 24.
Is equal to 21 divided by 8.

Expectation of Y square is equal to summation over y is equal to 1 to 2, y square into probability of Y is equal to y Is equal to summation over y is equal to 1 to 2, y square into 3 plus 6 into y whole divided by 24 Is equal to 1 square into 3 plus 6 into 1 whole divided by 24 plus 2 square into 3 plus 6 into 2 whole divided by 24 Is equal to 1 into 9 by 24 plus 4 into 15 by 24Is equal to 69 divided by 24 is equal to 23 divided by 8. Variance of X is equal to expectation of X square minus Expectation of X the whole square Is equal to 21 by 8 minus 37 by 24 whole square Is equal to zero point 2, 4, 8, 3. Variance of Y is equal to expectation of Y square minus expectation of Y the whole square Is equal to 23 by 8 minus 13 by 8 the whole square Is equal to zero point 2, 3 4, 4.

To find covariance of x, y, let us find expectation of X into Y
Expectation of X Y is equal to summation over x is equal to 1 to 2, summation over y is equal to 1 to 2 x into y into probability of X is equal to x intersection Y is equal to Y Is equal to summation over x is equal to 1 to 2, summation over y is equal to 1 to 2 x into y into x plus 3 into y divided by 24 Is equal to summation over x is equal to 1 to 2, x into summation over y is equal to 1 to 2, y into x plus 3 into y whole divided by 24 Is equal to summation over x is equal to 1 to 2, x into 1 into x plus 3 into 1 whole divided by 24 plus 2 into x plus 3 into 2 whole divided by 24 Is equal to summation over x is equal to 1 to 2, x into 3 into x plus 15 whole divided by 24 Is equal to 1 into 3 into 1 plus 15 whole divided by 24 plus 2 into 3 into 2 plus 15 whole divided by 24 Is equal to 60 divided by 24 Is equal to 5 by 2.

Hence, covariance is given by,
Covariance of X Y is equal to expectation of X into Y minus Expectation of X into Expectation of Y
Is equal to 5 divided by 2 minus 37 divided by 24 into 13 divided by 8 Is equal to minus zero point zero, zero 5, 2.

4. Exercise - 3

Let f of x, y is equal to 21 into x square into y cube, zero less than x less than y less than 1 and zero elsewhere be the joint probability density function of X and Y. Find the conditional mean and variance of X given Y is equal to y, where zero less than y less than 1.

Let us solve the problem as follows.
Given f of x, y is equal to 21 into x square into y cube, zero less than x less than y less than 1 and zero elsewhere. Marginal probability density function of Y is given by f Y of y is equal to integral from zero to y, f of x, y dx Is equal to 21 into y cube into integral from zero to y, x square dx Is equal to 7 into y power 6.

Therefore, the conditional probability density function of X given Y is given by
f of x given y is equal to f of x, y divided by f Y of y is equal to 21 into x square into y cube whole divided by 7 into y power 6 is equal to 3 into x square divided by y cube, where zero less than x less than y and zero less than y less than 1.

Conditional mean of X is
Expectation of X given y is equal y
Is equal to integral from zero to y, x into f of x given y dx Is equal to 3 divided by y cube into integral from zero to y, x cube dx

On integrating and simplifying we get 3 into y divided by 4, where zero less than y less than 1. To find variance first let us find the expectation of X square given Y is equal to y Is equal to integral from zero to y, x square into f of x given y dx Is equal to 3 divided by y cube into integral from zero to y x power 4 dx Is equal to 3 divided by y cube into y power 5 divided by 5 is equal to 3 divided by 5 into y square, where zero less than y less than 1. Therefore, Variance of X given Y is equal to y is equal to Expectation of of X square given Y is equal to y minus Expectation of X given Y is equal to y the whole square Is equal to 3 divided by 5 into y square minus 9 divided by 16 into y square Is equal to 3 divided by 80 y square, where zero less than y less than 1.

5. Exercise - 4

Let X and Y be two random variables each taking three values minus 1, 0, 1 and having the following joint probability distribution, find Variance of (Y given X is equal to minus1)
The table shows the values taken by the variables X and Y with their respective probabilities. That is when Y takes value minus 1, X take values minus 1, zero and 1 with respective probabilities zero, zero point 1, zero point 1. When Y takes value zero, X take values minus 1, zero and 1 with probabilities zero point 2, zero point 2 and zero point 2 respectively and When Y takes value 1, X take values, minus 1, zero and 1 with respective probabilities zero, zero point 1 and zero point 1. The numbers written in bold gives the row totals and column total and the grand total adds up to 1.

We solve the above problem as follows.

We know that Variance of (Y given X is equal to minus 1) is equal to Expectation of Y square given X is equal to minus 1 minus Expectation of Y given x is equal to minus 1 the whole square. Name it as 1.
Expectation of Y given x is equal to minus 1 is equal to summation over y, y into Probability of Y is equal to y and X is equal to minus 1 Is equal to minus 1 into zero plus zero into zero point 2 plus 1 into zero is equal to zero.
Expectation of Y square given x is equal to minus 1 is equal to summation over y, y square into Probability of Y is equal to y and X is equal to minus 1 Is equal to minus 1 square into zero plus zero square into zero point 2 plus 1 square into zero is equal to zero.

Therefore, on substituting the values in (1), we get, Variance of (Y given X is equal to minus 1) is equal to zero