1. Introduction

Welcome to the series of E-learning module on fitting of exponential curves. In this section, we will study different types of exponential curves, the method of fitting exponential curves to the given data using the method of least squares and predict the future values assuming the same trend continues.

By the end of this session, you will be able to:

- Explain different types of exponential curves
- Explain the fitting exponential curves using least square method
- Explain the Prediction of the future value assuming that the same pattern continues

Many processes in nature have exponential dependencies.

- The amplitude of a pendulum swinging in air decays along with the time
- The decrease in the temperature of an object along with time, i.e. initially warmer than its surroundings
- The growth of an initially small bacterial colony along with time

All these processes are well modeled by exponential relationships.

Depending on the pattern suggested by the scatter plot, there are many non-linear equations from which we can select a particular equation to fit a given data. In such cases, the original data, which is not in a linear form, can be reduced to linear form by simple transformation of variables. Usually, it will be done by taking logarithm of the function. It is valid since any number and its logarithm are monotonic functions. This is illustrated by considering the exponential curves.

There are different types of exponential curves.
Here we discuss the following two types of curves namely,
Y is equal to a into b power x and
Y is equal to a into e power b into x.

Let us consider the first one,
Y is equal to a into b power x.
By taking logarithm on both the sides, we get,
Log y is equal to log a plus x into log b
Implies U is equal to A plus B into x, where U is equal to Log Y, A is equal to log a and B is equal to log b.

Now let us find the normal equations by the method of least squares. Observe that the function U is a straight line, proceeding as we have done in the case of fitting linear curve or straight line, that is by minimizing the error, E is equal to summation u minus A minus B minus B into x whole square. That is by differentiating E with respect to A and B and then equating it to zero and simplifying we can write the normal equations. To prove that the

solutions obtained by the above expressions are minimum, we can show that the second derivative is positive.
Summation U is equal to n into A plus B into summation x
Summation U into x is equal to A into summation x plus B into summation x square.
Solving these two simultaneous equations, we can find the values of A and B and consequently we get,
a is equal to antilog of A and b is equal to antilog of B.
With these values of a and b, the exponential curve Y is equal to a into b power x, which best fits to the given set of n points.

Now let us consider the second type of exponential curve, Y is equal to a into e power b into x.
Taking logarithm on both the sides we get, log y is equal to log a plus b into x into log e
Is equal to log a plus b into log e into x, which is in the form U is equal to A plus B into x, a linear equation, where U is equal to log Y, A is equal to log a and B is equal to b into log e.
Now, let us find the normal equations by the method of least squares. Observe that the function U is a straight line, proceeding as we have done in the previous case, we get normal equations.

Summation U is equal to n into A plus B into summation x
Summation U into x is equal to A into summation x plus B into summation x square.
Solving these two simultaneous equations, we can find the values of A and B and consequently we get,
a is equal to antilog log of A and b is equal to B divided by log e.

With these values of a and b, the curve Y is equal to a into e power b into x, which best fits for the given set of n points.

2. **Illustration 1**

Fit an exponential curve of the form y is equal to a into b power x to the following data given below. Also, estimate y when x is equal to 10.

Figure 1

X	Y
1	1.0
2	1.2
3	1.8
4	2.5
5	3.6
6	4.7
7	6.6
8	9.1

The exponential curve fitted to the above data is y is equal to a into b power x
The normal equations are given by,
Summation U is equal to n into A plus B into summation x and
Summation U into x is equal to A into summation x plus B into summation x square, where U is equal to log Y, A is equal to log a and B is equal to log b.

To calculate the values of a and b, we construct the following table.

Figure 2

x	Y	U=logy	xU	x2
1	1.0	0.0000	0.0000	1
2	1.2	0.0792	0.1584	4
3	1.8	0.2553	0.7659	9
4	2.5	0.3979	1.5916	16
5	3.6	0.5563	2.7815	25
6	4.7	0.6721	4.0326	36
7	6.6	0.8195	5.7365	49
8	9.1	0.9590	7.6720	64
36	**30.5**	**3.7393**	**22.7385**	**204**

The first and 2nd columns in the table are written as it is in question.
Third column gives the logarithm of the values of the 2nd column. That is log 1.0 is equal to zero. Log 1.2 is equal to 0.0792, log 1.8 is equal to 0.2553 and so on.
The fourth column gives the product of first and third column. That is 1 into zero is equal to zero, 2 into 0.0792 is equal to zero point 1.584, 3 into 0.2553 is equal to 0.7659 and so on. The last column gives the squared values of the numbers in the first column. That is 1 square is equal to 1, 2 square is equal to 4 and so on. Once we find all the elements of different columns, we find totals of all the columns and denote it by bold numbers in the table.

Therefore, the normal equations can be written as follows.
3 point 7 three 9 three is equal to 8 A plus 36 B and 22 point 7 three 8 five is equal to 36 A plus 204 B
The above equations can be solved by using the Cramer's rule. Hence, consider the following determinants.
Delta is equal to determinant of 8, 36, 36, 204 is equal to 8 into 204 minus 36 into 36 is equal to 336.
Delta 1 is equal to determinant of 3 point 7 three 9 three, 36, 22 point 7 three 8 five 204 is equal to 3 point 7 three 9 three into 204 minus 22 point 7 three 8 five into 36 is equal to minus 55 point 7 six 8 eight.

Delta 2 is equal to determinant of 8, 3 point 7 three 9 three, 36, 22 point 7 three 8 five is equal to 8 into 22 point 7 three 8 five minus 36 into 3 point 7 three 9 three is equal to 47 point 2 nine 3 two.
Therefore, the constants in the normal equations

A is equal to delta 1 divided by delta is equal to minus 55 point 7 six 8 eight divided by 336 is equal to minus zero point 1 six 6.
And B is equal to delta 2 divided by delta is equal to 47 point 2 nine 3 two divided by 336 is equal to zero point 1 four zero 7 five.

Hence, constants of the curve y is given by, a is equal to antilog of A is equal to antilog of minus zero point 1 six 6 is equal to zero point 6 eight 2 three and
b is equal to antilog of B is equal to antilog of zero point 1 four zero 7 five is equal to 1 point 3 eight.
Therefore, the equation of the required curve is:
Y is equal to zero point 6 eight 2 three into 1 point 3 eight power x.

To estimate y when x is equal to 10, we substitute the value of x in the above equation. That is y is equal to zero point 6 eight 2 three into 1 point 3 eight power x is equal to zero point 6 eight 2 three into 1 point 3 eight power 10 Is equal to 17 point zero nine.

3. **Illustration 2**

Following table gives the profits of an enterprise for 5 consecutive years. Assuming that the data shows the exponential pattern of type Y is equal to a into b power x, estimate the profit for the year 2 thousand 13.

Figure 3

Year	2008	2009	2010	2011	2012
Profit (lakhs)	1.6	4.5	13.8	40.2	125.0

The exponential curve fitted to the above data is y is equal to a into b power x
The normal equations are given by,
Summation U is equal to n into A plus B into summation x and
Summation U into x is equal to A into summation x plus B into summation x square, where U is equal to log Y, A is equal to log a and B is equal to log b.

To calculate the values of a and b, we construct the following table.

Figure 4

Year	Profit (Y)	x	U=logy	xU	x2
2008	1.6	-2	0.2041	-0.4082	4
2009	4.5	-1	0.6532	-0.6532	1
2010	13.8	0	1.1399	0.0000	0
2011	40.2	1	1.6042	1.6042	1
2012	125.0	2	2.0969	4.1938	4
Total	185.1	0	5.6983	4.7366	10

If we choose given years as x values, then we may have very big numbers. Hence, to simplify our calculations we choose x such that summation x is equal to zero. That is x is equal to year minus 2 thousand 10, which is written in the third column. The fourth column is obtained by taking logarithm of Y. That is log 1 point 6 is equal to zero point 2 zero 4 one, log 4 point 5 is equal to zero point 6 five 3 two etc.
Fifth column is obtained by multiplying 3^{rd} and 4^{th} columns, that is minus 2 into zero point 2 zero 4 1 is equal to minus zero point 4 zero 8 2, minus 1 into zero point 6 five 3 two is equal to minus zero point 6 five 3 two and so on. And the last column is obtained by squaring the numbers of column 3. That is minus 2 square is equal to 4, minus 1 square is equal to 1 and so on.

Finally, we find the totals of all the columns.

Therefore, the normal equations can be written as follows.
The first equation is 5 point 6 nine 8 three is equal to 5 into A plus zero into B
Implies, A is equal to 5 point 6 nine 8 three divided by 5 is equal to 1 point 1 three 9 seven.
The second equation is 4 point 7 three 6 six is equal to zero into A plus 10 into B
Implies, B is equal to 4 point 7 three 6 six divided by 10 is equal to zero point four 7 three 6 six
Therefore, the constants of the exponential curve are given by,
a is equal to antilog of A is equal to antilog of 1 point 1 three 9 seven is equal to 13 point 7 nine 4 three and b is equal to antilog of B is equal to antilog of zero point four 7 three 6 six is equal to 2 point 9 eight.

The equation of required curve is, Y is equal to 13 point 7 nine 4 three into 2 point 9 eight power x.

To estimate Y for the year 2 thousand 13, we substitute the value of x is equal to 2 thousand 13 minus 2 thousand 10 is equal to 3 in the above equation.
That is, Y is equal to 13 point 7 nine 4 three into 2 point 9 eight power 3 is equal to 365 point zero 5 lakhs.

4. **Illustration 3**

Fit a curve of the form Y is equal to a into e power b into x for the following data regarding the production of a factory in tons.

Figure 5

Year	1990	1995	2000	2005	2010
Production	100	120	144	207.4	248.8

The exponential curve fitted to the above data is y is equal a into e power b into x.
The normal equations to find the constants of the data are:
Summation U is equal to n into A plus B into summation x
Summation U into x is equal to A into summation x plus B into summation x square.
Where, U is equal to log Y, A is equal to log a and B is equal to b into log e.
To calculate the values of a and b, we construct the following table. We do not consider the year as x. To simplify the calculations, we choose x as the sum of x is equal to zero. That is x=(year – 2000)/5

Figure 6

Year	**Production (Y)**	**x**	**U=logY**	**Ux**	**x2**
1990	100	-2	2.0000	-4.0000	4
1995	120	-1	2.0792	-2.0792	1
2000	144	0	2.1461	0	0
2005	207.4	1	2.3168	2.3168	1
2010	248.8	2	2.3976	4.7952	4
Total		**0**	**10.9397**	**1.0328**	**10**

The third column gives the values taken by x for different years. The fourth column is calculated by finding the logarithm of the production, that is Y which are given in the 2nd column. The fifth column is obtained by multiplying the elements in the 3rd and 4th column. And the last column is obtained by finding the squares of numbers in the third column. Once we calculate all the values, the totals of the columns, which are necessary are found and written in bold letters in the table.

Therefore, the normal equations can be written as follows.
The first normal equation is 10 point 9 three 9 seven is equal to 5 into A plus zero into B, implies, A is equal to 10 point 9 three 9 seven divided by 5 is equal to 2 point 1 eight 7 nine.
The second normal equation is 1 point zero 3 two 8 is equal to zero into a plus 10 into B, implies B is equal to 1 point zero 3 two 8 divided by 10 is equal to zero point 1 zero 3 three.
Therefore, the constants of the curve, a is equal to antilog of A is equal to antilog of 2 point 1

eight 7 nine is equal to 154 point 1
b is equal to B divided by log e is equal to zero point 1 zero 3 three divided by zero point 4 three 4 three is equal to zero point 2 three 7 four.

Hence, the required exponential curve is given by, Y is equal to 154 point 1 into e power zero point 2 three 7 four into x.

5. **Illustration 4**

Fit an equation of the form Y is equal to a into b power x for the following data.

Figure 7

x	Y
0.25	93
0.50	71
0.75	63
1.00	54
1.25	48
1.50	38
1.75	29
2.0	26
2.25	22

The exponential curve fitted to the above data is Y is equal to a into b power x. The normal equations are given by:
summation U is equal to n into A plus B into summation x
Summation U into x is equal to A into summation x plus B into summation x square.
Where, U is equal to log y, A is equal to log a and B is equal to log b.

To calculate the values of a and b, we construct the following table. We do not consider the x values as it is. We write x = (x-1.25)/0.25 so that we get sum of x is zero.

Figure 8

x	y	x	U	Ux	X2
0.25	93	-4	1.9685	-7.8739	16
0.50	71	-3	1.8513	-5.5538	9
0.75	63	-2	1.7993	-3.5987	4
1.00	54	-1	1.7324	-1.7324	1
1.25	48	0	1.6812	0.0000	0
1.50	38	1	1.5798	1.5798	1
1.75	29	2	1.4624	2.9248	4
2.00	26	3	1.4150	4.2449	9
2.25	22	4	1.3424	5.3697	16
Total		0	14.8323	-4.6396	60

The first two columns denote the values given in the problem. The new values of x are written in the column 3 of the table. 4th column is obtained by taking logarithm of the values of the 2nd column. The fifth column is obtained by multiplying the numbers of the 3rd and 4th columns and the last column is obtained by finding the square of the values of the 3rd column. Then the required totals of different columns are found.

Therefore, the normal equations can be written as follows.
The first normal equation is given by, 14 point 8 three 2 three is equal to 9 into A plus zero into B, implies, A is equal to 14 point 8 three 2 three divided by 9 is equal to 1 point 6 four 8 zero
The second normal equation is given by,
Minus 4 point 6 three 9 six is equal to zero into A plus 60 into B
Implies, B is equal to Minus 4 point 6 three 9 six divided by 60 is equal to minus zero point zero 7 seven 3.
Therefore, the constants are given by,
a is equal to antilog of A is equal to antilog of 1 point 6 four 8 zero is equal to 44 point 4 six three and b is equal to antilog of B is equal to antilog of minus zero point zero 7 seven 3 is equal to zero point 8 three 7.
Therefore, the equation is given by, Y is equal to 44 point 4 six 3 into zero point 8 three 7 power x, where x is equal to x minus one point 25 divided by zero point 25.

Here's a summary of our learning in this session, where we understand:

- The different types of exponential curves
- The fitting of exponential curves using the method of least squares
- The prediction of future value assuming that the same pattern continues