1. Introduction

Welcome to the series of E-learning modules on exponential distribution. Here we have discussed about the moments of exponential distribution, moment generating function, cumulant generating function and nature of the distribution. We have also stated and proved memory-less property of the distribution.

By the end of this session, you will be able to:

- Describe Exponential Distribution
- Understand Moment generating function and hence mean and variance
- Explain Cumulant Generating Function
- Understand nature of the distribution
- Explain median , mode and memoryless property

A continuous random variable X is said to have an exponential distribution with parameter theta if its probability density function is given by, f of (x) is equal to theta into e power minus x into theta, where x is greater than zero.
And we write X follows exponential theta.
Observe that, the probability density function contains the function of x, which is in power of exponential. Hence we can obtain the mean and variance of the distribution using moment generating function rather than finding it directly.

Hence let us find moment generating function.
If X follows exponential distribution with parameter theta, then moment generating function is given by
M X of t is equal to expectation of e to the power t into X
Is equal to integral from zero to infinity, e to the power t into x into f of x d x
Is equal to theta into e to the power t into x into e to the power minus x into theta d x
Is equal to theta into integral from zero to infinity e to the power minus of theta minus t into x d x
On integrating the function we get
Theta into e to the power minus x into theta minus t divided by minus of theta minus t, ranges from zero to infinity
Is equal to theta divided by theta minus t
Is equal to one divided by one minus t by theta
Is equal to one minus t by theta whole power minus one, provided mod t by theta is less than one.

Note that, We can obtain moments from moment generating function by

- Differentiating M X of (t) with respect to t at t is equal to zero
- Equating the coefficient of t power r divided by r factorial in M X of (t)

First let us find mean and variance by differentiating and then by equating the coefficient.

2. Mean and Variance from MGF

Let us find mean from moment generating function using differentiation method.
We know that M X of t is equal to [one minus (t divided by theta] whole power minus one.
Expectation of X is equal to d by d x of M X of t, at t is equal to zero
Is equal to d by d x of one minus t divided by theta whole power minus one at t is equal to zero
Is equal to minus of one minus t divided by theta whole power minus two into one by theta at t is equal to zero
Is equal to one divided by theta.
Hence mean is reciprocal of the parameter.

Now let us find variance.
Variance of the distribution is given by, Variance of (X) is equal to Expectation of (X square) minus [Expectation of (X)] the whole square.
First let us find Expectation f (X square)
Expectation of x square is equal to d square by d x square M X of t at t is equal to zero
Is equal to one by theta into d by d x of one minus t by theta whole power minus two, at t is equal to zero
Is equal to one by theta into minus two into one minus t by theta whole power minus three into minus one by theta at t is equal to zero
Is equal to two divided by theta square.

Hence variance of X is equal to two divided by theta square minus one divided by theta the whole square
Is equal to one divided by theta square.

Hence observe that if X follows exponential theta, the mean is equal to one by theta and variance is equal to one by theta square
That is variance is equal to one by theta into one by theta is equal to mean by theta

Therefore,
Variance is greater than mean if zero less that theta less than one
Variance is equal to mean if theta is equal to one and
Variance is less than theta if theta is greater than one
Hence for exponential distribution,
Variance is greater or equal to less than mean for different values of the parameters.

Now let us find mean and variance by equating the coefficient of t power r divided by r factorial.
Consider the moment generating function
M X of t is equal to one minus t divided by theta whole power minus one
Using geometric expansion of infinite series we get,
one plus t divided by theta plus t divided by theta the whole square plus etc. and so on

Now Expectation of (X power r) is equal to coefficient of t power r divided by r factorial.
Therefore Expectation of (X) is equal to coefficient of t divided by one factorial is equal to one

divided by theta.
Expectation of (X square) is equal to Coefficient of t square divided by two factorial is equal to two divided by theta square.
Hence variance of x is equal to Expectation of (X square) minus [Expectation of (X)] the whole square
Is equal to two divided by theta square minus one minus theta the whole square
Is equal to one divided by theta square.

3. Cumulant Generating Function, Nature and Median of the Distribution

Now let us find the Cumulant generating function the exponential distribution with parameter theta.
We know that Cumulant generating function is given by
K X of (t) is equal to log (M X of (t)) is equal to log [one minus (t divided by theta] the whole power minus one.
Is equal to minus one into log [one minus (t divided by theta)]
Using logarithm expansion we get,
Minus one into t divided by theta minus t divided by theta the whole square divided by two minus t divided by theta the whole cube divided by three minus t divided by theta the whole power four divided by four minus etc.
Is equal to t divided by theta plus t divided by theta the whole square divided by two plus t divided by theta the whole cube divided by three plus t divided by theta the whole power four divided by four plus etc.

Hence we get
K one is equal to coefficient of t divided by one factorial is equal to one divided by theta which is equal to mu one dash, mean of the distribution
K two is equal to coefficient of t square divided by two factorial is equal to one divided by theta square is equal to mu two, the variance of the distribution
K three is equal to coefficient of t cube divided by three factorial is equal to two divided by three factorial is equal to two divided by theta cube is equal to mu three.
K four is equal to coefficient of t power divided by four factorial is equal to six divided by theta power four
But mu four is equal to K four plus three into mu two square is equal to six divided by theta power four plus three divided by theta power four is equal to nine divided by theta power four

Now we can find the nature of the distribution
Let us find coefficient of skewness
Gamma one is equal to mu three divided by mu two to the power three by two is equal to two by theta cube divided by one by theta square the whole to the power three by two
Is equal to two, which is greater than zero
Coefficient of kurtosis is given by,
Beta two is equal to mu four divided by mu two square is equal to nine by theta power four divided by one by theta square is equal t nine, which is greater than three.
Hence exponential distribution is positively skewed and has leptokurtic curve.

Now let us find median of the distribution.
Median divides the distribution into two equal parts. Hence if M is the median of the distribution then
Integral from zero to M f of x d x is equal to integral from M to infinity f of x d x is equal to half.

We can consider any one of the integral. Let us consider the first.

By substituting for f of x, we get
Theta into integral from zero to M 'e' to the power minus x into theta d x is equal to half
Implies minus of 'e' to the power minus x into theta, ranges from zero to M is equal to half
Implies e to the power minus M into theta is equal to half
Implies, M is equal to log two to the base 'e' two divided by theta.

4. Mode, Memoryless Property

Let us discuss about mode of the distribution.
Observe that exponential distribution is strictly a decreasing function, f of (x) is maximum at x is equal to zero. Hence mode of the distribution is zero.

Observe that mean of an exponential distribution is the reciprocal of the parameter.
Suppose the mean of the distribution is theta, the parameter of the distribution is one divided by theta. Hence we can define an exponential distribution with mean theta as follows.

Now let us define the memory less property of the exponential distribution.
Exponential distribution 'lacks memory', that is if X has an exponential distribution, then for every constant 'a' greater than or equal to zero, one has
Probability that (Y is less than or equal to x given that X is greater than or equal to 'a') is equal to Probability that (X is less than or equal to x) for all x, where Y is equal to X minus 'a'

Let prove this property as follows.
We have Probability that (Y is less than or equal to x intersection X is greater than or equal to 'a') is equal to Probability that (X minus 'a' less than or equal to x intersection X greater than or equal to 'a')
Is equal to Probability that (X is less than or equal to x plus 'a' intersection X is greater than or equal to 'a')
Is equal to Probability that ('a' is less than or equal to X less than or equal to 'a' plus x)
Is equal to theta into integral from 'a' to 'a' plus x, e to the power minus theta into x dx
Is equal to e power minus 'a' into theta into one minus 'e' to the power minus theta into x

And
Probability that X is greater than equal to 'a' is equal to theta into integral from 'a' to infinity, e to the power minus theta into x d x is equal to e to the power minus 'a' into theta.
Hence probability that Y less than or equal to X given X greater than or equal to 'a'
Is equal to probability that Y less than or equal to x intersection X greater than or equal to 'a' divided by probability that X greater than or equal to 'a'
Is equal to e to the power minus 'a' into theta into one minus e power minus theta into x divided by e to the power minus 'a' into theta
Is equal to one minus e to the power minus theta into x
Also probability that X less than or equal to x is equal to theta into integral from zero to x, e power minus theta into x d x is equal to one minus e to the power minus theta into x
Hence probability that Y less than or equal to x given X greater than or equal to 'a' is equal to Probability that X less than or equal to x
That is exponential distribution lacks memory.

A random variable X is said to follow exponential distribution with mean theta if its probability density function is given by f of x is equal to one by theta into e to the power minus x divided by theta where x is greater than zero.
Therefore, Mean is equal to theta.
And variance is equal to two into theta square.

If X I, where I is equal to one two etc till n are independent exponential random variables with parameter theta I. Let Z is equal to minimum of X one , X two, etc.tll i X n, then Z follows exponential distribution with parameter summation theta i.

Let us prove this result as follows.
Since x i's are independent exponential random variables with parameter theta i, probability density function is given by f of (x i) is equal to theta i into e to the power minus x into theta i, where x i is greater than zero.
To prove above result first we consider
Probability that (Z greater than x) is equal to Probability of [minimum of (X one, etc., X n) is greater than x]
Is equal to Probability that [X one is greater than x, X two is greater than x etc., X n is greater than x]

Since the random variables are independent,
Probability that (Z greater than x)
Is equal to Probability that [X one greater than x] into Probability that [X two greater than x] into etc into Probability that [X n greater than x]
Is equal to (e to the power minus theta one into x) into (e power minus theta two into x) into etc. into (e to the power minus theta n into x),
Is equal to product from I is equal to one to n, e to the power minus theta I into x
Is equal to e to the power minus summation over I is equal to one to n theta I into x, which is the moment generating function of exponential distribution with parameter summation theta i.
Hence by uniqueness theorem of moment generating function, Z follows exponential distribution with parameter (summation theta i).

5. Illustrations Contd.

Consider the following illustration.
Suppose that the amount of time one spends in a bank is exponentially distributed with mean ten minutes, what is the probability that a customer will spend more than fifteen minutes in the bank? What is the probability that a customer will spend more than fifteen minutes in the bank given that he is still in the bank after ten minutes?

Let us solve the above problem as follows.
Let X denote amount of time one spends in a bank. Hence X follows exponential distribution with parameter theta.
Mean is ten minutes. Hence theta is equal to one divided by ten is equal to zero point one

Therefore probability density function is given by; f of (x) is equal to theta into e to the power minus x into theta where x is greater than zero.
By substituting theta, we get f of (x) is equal to (zero point one) into e to the power minus zero point one into x, where x greater than zero,
Consider
Probability that (Customer will spend more than fifteen minutes)
Is equal to Probability that (X greater than fifteen)
Is equal to zero point one into integral from fifteen to infinity, e to the power minus zero point one into x d x
Is equal to e power minus zero point one into fifteen
Is equal to zero point two, two

Now we need to find the probability that customer will spend more than fifteen minutes in the bank given he is still in the bank after ten minutes
That is, Probability that (X greater than fifteen given X greater than ten)
Using memory less property we can write,
Probability that (X greater than fifteen given X greater than ten)
Is equal to Probability that (X greater than five)
Is equal to e to the power minus zero point one into five is equal to zero point six, zero four.

Illustration 2
Suppose we are told that, on average, there are two hits per minute on a specific web page. What is the probability that we have to wait at most forty seconds to observe the first hit?
Also find How long do we have to wait at most, to observe a first hit with a probability of zero point nine?

Let us solve above problem as follows.
Let X denotes the number of hits per minute on a specific web page. Here X follows exponential distribution with parameter theta.
Average number of hits is two. Hence theta is equal to half is equal to zero point five. Probability density function of exponential distribution is given by; f of (x) is equal to theta into e to the power minus x into theta, where x is greater than zero.

By substituting theta, we get f of (x) is equal to (zero point five) into e power minus zero point

five into x, where x is greater than zero.
Since we are working in time units of minutes, we need to express the forty seconds above as forty divided by sixty is equal to two divided by three minutes.
Thus, we compute the probability that we have to wait at most two divided by three minutes to observe the first hit: that is
probability that X less than or equal to two divided by three
Is equal to zero point five into integral from zero to two by three e to the power minus zero point five into x d x
Is equal to zero point five into e to the power minus zero point five into x divided by minus zero point five, ranges from zero to three by two.
Is equal to one minus e to the power minus zero point five into two by three
Is equal to zero point seven, three, six.

This is the reverse of what we have computed so far, because here we want to find'‘t’, for which Probability that (Y is less than or equal to‘t’) is equal to zero point nine.
Probability that (X less than or equal to t) is equal to zero point nine
That is zero point five into integral from zero to t, e to the power minus zero point five into x d x is equal to zero point nine.
Implies one minus e to the power minus two into t is equal to zero point nine
Implies e to the power two into t is equal to zero point one
That is, t is equal to minus zero point five into natural logarithm of zero point one
Nearly equal to one point one, five minutes, that’s approximately sixty nine seconds.

Here’s a summary of our learning in this session where we have :

- Understood about the exponential distribution
- Described Moment generating function and hence mean and variance
- Explained Cumulant generating function
- Understood Nature of the distribution.
- Explained median , mode and memoryless property