# Frequently Asked Questions

1. Define exponential distribution.
   **Answer:**
   A continuous random variable X is said to have an exponential distribution with parameter θ if its pdf is given by, $f(x)=\theta e^{-x\theta}$, x>0.
   And we write X~exp(θ)

2. Define exponential distribution with mean θ
   **Answer:**
   A continuous random variable X is said to have an exponential distribution with mean θ if its pdf is given by, $f(x) = \frac{1}{\theta} e^{-x/\theta}, x > 0$

3. Obtain mgf of exponential distribution.
   **Answer:**
   If X~exp (θ), then mgf is given by,
   $$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx = \theta\int_0^\infty e^{tx}e^{-x\theta}dx$$
   $$= \theta\int_0^\infty e^{-(\theta-t)x}dx = \theta\left|\frac{e^{-x(\theta-t)}}{-(\theta-t)}\right|_0^\infty = \frac{\theta}{\theta-t} = \frac{1}{1-t/\theta} = (1-t/\theta)^{-1}$$
   provided |t/θ|<1

4. Derive mean of the distribution from mgf using differentiation.
   **Answer:**
   We know that $M_X(t)=[1-(t/\theta)]^{-1}$
   $$E(X) = \frac{d}{dx}M_X(t)\Big|_{t=0} = \frac{d}{dx}\left(1-\frac{t}{\theta}\right)^{-1}\Big|_{t=0} = -\left(1-\frac{t}{\theta}\right)^{-2}\times\left(\frac{1}{\theta}\right)\Big|_{t=0} = \frac{1}{\theta}$$

5. Obtain variance of the exponential distribution from mgf using the method of differentiation.
   **Answer:**
   Variance of the distribution is given by, $V(X) = E(X^2)-[E(X)]^2$
   $$E(X) = \frac{d}{dx}M_X(t)\Big|_{t=0} = \frac{d}{dx}\left(1-\frac{t}{\theta}\right)^{-1}\Big|_{t=0} = -\left(1-\frac{t}{\theta}\right)^{-2}\times\left(\frac{1}{\theta}\right)\Big|_{t=0} = \frac{1}{\theta}$$
   $$E(X^2) = \frac{d^2}{dx^2}M_X(t)\Big|_{t=0} = \frac{1}{\theta}\frac{d}{dx}\left(1-\frac{t}{\theta}\right)^{-2}\Big|_{t=0}$$
   $$= \left(\frac{1}{\theta}\right)\left\{-2\times\left(1-\frac{t}{\theta}\right)^{-3}\times\left(-\frac{1}{\theta}\right)\Big|_{t=0}\right\} = \frac{2}{\theta^2}$$
   $$V(X) = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}$$

6. Write the relationship between mean and variance of exponential distribution.

**Answer:**

If X~exp (θ), then mean=1/θ and variance = $1/\theta^2$

I.e. Variance= (1/θ) (1/θ) =mean/θ

Therefore Variance>Mean, if 0<θ<1

Variance=Mean if θ=1 and

Variance<Mean if θ>1

Hence for the exponential distribution,

Variance>,= or < Mean for different values of the parameter.

7. Obtain the expression for mean and variance by equating the coefficient of $t^r/r!$ in mgf.

**Answer:**

Let us find mean and variance by equating the coefficient of $t^r/r!$. Consider the mgf,

$$M_X(t) = \left(1 - \frac{t}{\theta}\right)^{-1} = 1 + t/\theta + (t/\theta)^2 + \ldots$$

Now E ($X^r$) =coefficient of $t^r/r!$.

Therefore E(X)=coefficient of t/1!=1/θ

$E(X^2)$=Coefficient of $t^2/2!=2/\theta^2$

Hence V(X)= $E(X^2)-[E(X)]^2$

$= 2/\theta^2-(1/\theta)^2$

$= 1/\theta^2$

8. Find Cumulant generating function.

**Answer:**

We know that Cumulant generating function is given by

$K_X(t)=\log(M_X(t))=\log[1-(t/\theta)]^{-1}$

$=-1\log[1-(t/\theta)]$

$$= -1\left[t/\theta - \frac{(t/\theta)^2}{2} - \frac{(t/\theta)^3}{3} - \frac{(t/\theta)^4}{4} - \ldots\right]$$

$$= t/\theta + \frac{(t/\theta)^2}{2} + \frac{(t/\theta)^3}{3} + \frac{(t/\theta)^4}{4} + \ldots$$

9. Obtain the first four Cumulants from cgf.

**Answer:**

$K_1$=coefficient of t/1!=1/θ

$K_2$=coefficient of $t^2/2!=1/\theta^2$

$K_3$=coefficient of $t^3/3!=2/\theta^3=\mu_3$

$K_4$=coefficient of $t^4/4!=6/\theta^4$

10. Comment on the nature of an exponential distribution.

**Answer:**

Let us find coefficient of skewness

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2/\theta^3}{(1/\theta^2)^{3/2}} = 2 > 0$$

Coefficient of kurtosis is given by,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9/\theta^4}{(1/\theta^2)^2} = 9 > 3$$

Hence exponential distribution is positively skewedand has leptokurtic curve.

11. Find median of an exponential distribution.

**Answer:**

Median divided the distribution into two equal parts. Hence if M is the median of the distribution then $\int_0^M f(x)dx = \int_M^\infty f(x)dx = \frac{1}{2}$

We can consider any one of the integral. Let us consider the first.

$$\theta\int_0^M e^{-x\theta}dx = \frac{1}{2}$$

$$\Rightarrow -\left|e^{-x\theta}\right|_0^M = \frac{1}{2}$$

$$\Rightarrow e^{-M\theta} = \frac{1}{2}$$

$$\Rightarrow M = \log_e(2)/\theta$$

12. What is the mode of an exponential distribution?

**Answer:**

Observe that exponential distribution is strictly a decreasing function, f(x) is maximum at x=0. Hence mode of the distribution is zero.

13. State and prove memoryless property of an exponential distribution.

**Answer:**

Exponential distribution 'lacks memory', i.e. if X has an exponential distribution, then for every constant a≥0, one has

P (Y≤x/X≥a) =P (X≤x) for all x, where Y=X-a

Proof:

We have P (Y≤x ∩ X≥a) =P(X-a≤x ∩ X≥a)

=P (X≤x+a ∩ X≥a)

=P (a≤X ≤a+x)

$$= \theta\int_a^{a+x} e^{-\theta x}dx = e^{-a\theta}(1 - e^{-\theta x})$$

And $P(X \ge a) = \theta\int_a^\infty e^{-\theta x}dx = e^{-a\theta}$

P (Y≤x/X≥a) $= \frac{P(Y \le x \cap x \ge a)}{P(X \ge a)} = \frac{e^{-a\theta}(1 - e^{-\theta x})}{e^{-a\theta}} = (1 - e^{-\theta x})$

Also $P(X \le x) = \theta\int_0^x e^{-\theta x}dx = 1 - e^{-\theta x}$

Hence,

P (Y≤x/X≥a) =P (X≤x)

i.e. exponential distribution lacks memory.

14. Show that If $X_i$, i = 1, 2, n, are independent exponential r.v.s with parameter θi. Let Z = min($X_1$... Xn), then Z~exp($\Sigma\theta_i$).

**Answer:**

Since $X_i$'s are independent exponential r.v.s with parameter θi, pdf is given by

$f(x_i) = \theta_i e^{-x\theta_i}$, $x_i > 0$

To prove above result first we consider

P (Z>x) =P [min($X_1$... Xn)>x]

$= P[X_1>x, X_2>x \ldots X_n>x]$

Since the random variables are independent,

$P(Z>x) = P[X_1>x]P[X_2>x]\ldots P[X_n>x]$

$= e^{-\theta 1x}\, e^{-\theta 2x} \ldots e^{-\theta nx} = \prod_{i=1}^{n} e^{-\theta_i x} = e^{-(\sum_{i=1}^{n}\theta_i)x}$, which is the mgf of exponential distribution with parameter $\Sigma\theta_i$. Hence by uniqueness theorem of mgf, Z ~ exp($\Sigma\theta_i$)

15. Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes, what is the probability that a customer will spend more than 15 minutes in the bank? What is the probability that a customer will spend more than 15 minutes in the bank given that he is still in the bank after 10 minutes?

**Answer:**

Let X denote amount of time one spends in a bank. Hence X~exp(θ)

Mean is 10 minutes. Hence θ=1/10=0.1

Therefore pdf is given by, $f(x)=\theta e^{-x\theta}$, x>0.

By substituting θ, we get $f(x) = (0.1)e^{-0.1x}$, x>0

Consider

P(Customer will spend more than 15 minutes)

$= P(X > 15) = 0.1\int_{15}^{\infty} e^{-0.1x}dx = e^{-0.1\times 15} = 0.22$

Now we need to find the probability that customer will spend more than 15 minutes in the bank given he is still in the bank after 10 minutes

i.e., P(X > 15|X > 10)

Using memory less property we can write,

$P(X > 15|X > 10) = P(X > 5) = e^{-0.1x5} = 0.604$