1. Introduction

Welcome to the series of E-learning modules on measures and related results of multiple and partial correlation and regression in three variables. Here we have discussed different results related to multiple and partial correlation coefficients.

By the end of this session, you will be able to:

- Explain the related measures and results of multiple and partial correlation coefficients

In previous modules, we have studied about Partial and multiple correlation and regression. Now, let us discuss about some of measures and related results of the multiple and partial correlation coefficient.

Show that the correlation coefficient between the residuals (X 1 point 2, 3) and (X2 point 1, 3) is equal and opposite to that between (X1 point 3) and (X2 point 3).
We prove this result as follows.
The correlation coefficient between (X1 point 2, 3) and (X2 point 1, 3) is given by,
(Covariance of X1 point 2, 3) and (X2 point 1, 3) divided by (sigma 1 point 2, 3) into (sigma 2 point 1, 3)
Is equal to (summation X1 point 2, 3) into (X2 point 1, 3) divided by (N into sigma 1 point 2, 3) into (sigma 2 point 1, 3)
Is equal to (1 by N) into (summation X2 point 1, 3) into (X1 minus b1, 2 point 3) into (X2 minus b1, 3 point 2) into (X3) divided by (sigma 1 point 2, 3) into (sigma 2 point 1, 3).

Is equal to (minus b1, 2 point 3) into (summation X2 point 1, 3) into (X2) divided by (N into sigma 1 point 2, 3) into (sigma 2 point 1, 3)
Is equal to (minus b1, 2 point 3) into (summation X2 point 1, 3 square) divided by (N into sigma 1 point 2, 3) into (sigma 2 point 1, 3)
Is equal to (minus b1, 2 point 3) into (sigma 2 point 1, 3) divided by (sigma 1 point 2, 3) is equal to (minus b1, 2 point 3) into (sigma 2) into (square root of omega) by (omega 2, 2) divided by (sigma 1) into (square root of omega) by (omega 1, 1).
Where omega is equal to determinant of (1) (r1, 2) (r1, 3) (r2, 1) (1) (r2, 3) (r3, 1) (r3, 2) (1).

(Omega 1, 1) is equal to determinant of (1) (r2, 3) (r3, 2) (1) is equal to (1 minus r2, 3 square) and
(Omega 2, 2) is equal to determinant of (1) (r1, 3) (r3, 1) (1) is equal to (1 minus r 3, 1 square).
Therefore, correlation coefficient between (X1 point 2, 3) and (X2 point 1, 3) is equal to (minus b1, 2 point 3) into (sigma 2) by (sigma 1) into (square root of 1 minus r2, 3 square) divided by (1 minus r1, 3 square) is equal to (minus b1, 2 point 3) into (sigma 2 point 3) divided by (sigma 1 point 3).
(Since sigma 2 point 3 square is equal to sigma 2 square into 1 minus r2, 3 square and sigma 1 point 3 square is equal to sigma 1 square into 1 minus r1, 3 square.)

Hence, the coefficient of correlation, (r of X1 point 2, 3) and (X2 point 1, 3) is equal to (minus covariance of X1 point 3), (X2 point 3) divided by (sigma 2 point 3 square) into (sigma 2 point 3) divided by (sigma 1 point 3)

Is equal to (minus covariance of X1 point 3), (X2 point 3) divided by (sigma 2 point 3) into (sigma 1 point 3) is equal to (minus r X1 point 3), (X2 point 3).
Hence the result.

2. Result 2

Show that if (X3 is equal to a) into (X1 plus b into X2), the three partial correlations are numerically equal to unity, (r1, 3 point 2) having the sign of (a, r2, 3 point 1), the sign of (b and r1, 2 point 3), the opposite sign of (a by b).
Proof of this result is as follows.
Here, we may regard X3 as dependent on X1 and X2 which may be taken as independent variables. Since X1 and X2 are independent, they are uncorrelated.

Thus, (r X1, X2) is equal to zero implies, covariance of (X1, X2 is equal to zero).
Variance of X3 is equal to Variance of(a into X1) plus (b into X2) is equal to (a square) into variance of X1 plus (b square) into variance of X2 plus (2 into a into b into covariance of X1, X2)
Is equal to (a square) into (sigma 1 square plus b square) into (sigma 2 square).
Where, variance of X1 is equal to (sigma 1 square) and variance of X2 is equal to (sigma 2 square). Also, (X1 into X3) is equal to (X1 into a into x1 plus b into X2)
Is equal to (a) into (X 1 square) plus (b) into (x1) into (X2).

Assuming that Xi's are measured from their means, on taking expectations of both sides, we get,
(Covariance of X1, X3) is equal to (a into sigma 1 square) plus (b into covariance of X1, X2) is equal to (a into sigma 1 square) Therefore, (r1, 3) is equal to (covariance of X1, X3) divided by square root of variance of X1 into variance of X3 is equal to (a into sigma 1 square) divided by square root of (sigma 1 square) into (a square) into (sigma 1 square) plus (b square) into (sigma 2 square) is equal to (a into sigma 1) divided by k.

Where (k square) is equal to (a square) into (sigma 1 square) plus (b square) into (sigma 2 square).
Similarly we will get, (r2, 3) is equal to (covariance of X2, X3) divided by (variance of X2) into (variance of X3)
Is equal to (b) into (sigma 2 square) divided by square root of (sigma 2 square) into (a square) into (sigma 1 square) plus (b square) into (sigma 2 square).
Is equal to (b) into (sigma 2) divided by (K).

Hence, (r1, 3 point 2) is equal to (r1, 3 minus r1, 2) into (r3, 2) divided by square root of (1 minus r1, 2 square) into (1 minus r3, 2 square) is equal to (a) into (sigma 1) divided by (k) into (k) divided by (square root of d square) minus (b square) into (sigma 2 square) is equal to (a) into (sigma 1) divided by (square root of a square) into (sigma 1 square) is equal to (a) into (sigma 1) divided by modulus of (a) into (sigma 1) is equal to (plus or minus 1), according as a is positive or negative. Hence r1, 3 point 2 has the same sign as a.
Again,
(r2, 3 point 1) is equal to (r2, 3 minus r2, 1) into (r3, 1) divided by square root of (1 minus r2, 1 square) into (1 minus r3, 1 square).
is equal to b into sigma 2 divided by k into k by square root of k square minus a square into sigma 1 square
is equal to b into sigma 2 divided by square root b square into sigma 2 square
is equal to b into sigma 2 divided by modulus of b into sigma 2

is equal to (plus or minus 1)

According as (b is positive or negative). Hence, (r2, 3 point 1) has the same sign as b.
(Now r1, 2 point 3) is equal to (r1, 2 minus r1), (3 into r2, 3) divided by square root of (1 minus r1, 3 square) into (1 minus r2, 3 square) is equal to (minus a) into (sigma 1) by (k) into (b) into (sigma 2) by (k) into (k square) divided by square root of (k square minus a square) into (sigma 1 square) into (k square minus b square) into (sigma 2 square)
is equal to (minus a) into (b) into (sigma 1) into (sigma 2) divided by square root of (a square) into (sigma 1 square) into (b square) into (sigma 2 square).
is equal to (minus a) into (b) divided by square root of (a square) into (b square).
Which can be written as,
(Minus a) by (b) divided by square root of (a square) by (b square)
Is equal to (minus a) by (b) divided by modulus of (a by b)
Is equal to (minus or plus 1) according as (a by b) is positive or negative. Hence, (r1, 2 point 3) has the sign opposite to that of (a by b).

3. Result 3

If (r1, 2) and (r1, 3) are given, show that (r2, 3) must lie in the range (r1, 2) into (r1, 3) plus or minus (1) minus (r1, 2 square) minus (r1, 3 square) plus (r1, 2 square) into (r1, 3 square power half).
If (r1, 2) is equal to (k), and (r1, 3) is equal to (minus k), show that (r2, 3) will lie between (minus 1 and 1 minus 2) into (k square).

To prove this result let us consider (r1, 2 point 3 square).
That is, (r1, 2 point 3 square) is equal to (r1, 2) minus (r1, 3) into (r2, 3) divided by square root of (1 minus r1, 3 square) into (1 minus r2, 3 square), the whole square is less than or equal to 1.
Therefore, (r1, 2) minus (r1, 3) into (r2, 3) whole square is less than or equal to (1 minus r1, 3 square) into (1 minus r2, 3 square).
Implies, (r1, 2 square) plus (r1, 3 square) into (r2, 3 square) minus (2) into (r1, 2 into (r1, 3) into (r2, 3) is less than or equal to 1 minus (r1, 3 square) minus (r2, 3 square) plus (r1, 3 square) into (r2, 3 square)
Implies, (r1, 2 square) plus (r1, 3 square) plus (r 2, 3 square) minus (2) into (r 1, 2) into (r1, 3) into (r2, 3) is less than or equal to 1.

This condition holds for consistent values of (r1, 2) (r1, 3) and (r2, 3). And hence, the above inequality can be rewritten as,
(r2, 3 square) minus (2) into (r1, 2) into (r1, 3) into (r2, 3) plus (r1, 2 square) plus (r1, 3 square) minus (1) is less than or equal to zero
Hence, for given values of (r1, 2) and (r1, 3) (r2, 3) must lie between the roots of the quadratic (in r2, 3) equation,
(r2, 3 square) minus (2) into (r1, 2) into (r1, 3) into (r2, 3) plus (r1, 2 square) plus (r1, 3 square) minus (1) is equal to zero.
Which are given by, (r2, 3) is equal to (r1, 2) into (r1, 3) plus or minus square root of (r1, 2 square) into (r1, 3 square) minus (r1, 2 square) plus (r1, 3 square) minus 1.

Hence (r1, 2) into (r1, 3) minus square root of (1) minus (r1, 2 square) minus (r1, 3 square) plus (r1, 2 square) into (r1, 3 square) is less than or equal to r2, 3 is less than or equal to (r1, 2) into (r1, 3) plus square root of (1) minus (r1, 2 square) minus (r1, 3 square) plus (r1, 2 square) into (r1, 3 square), name it as equation star.
In other words, (r2, 3) must lie in the range, (r1, 2) into (r1, 3) plus or minus square root of (1) minus (r1, 2 square) minus (r1, 3 square) plus (r1, 2 square) into (r1, 3 square).
In particular, if (r1, 2) is equal to (k) and (r1, 3) is equal to (minus k), we get from star,
(Minus k square) minus square root of (1) minus (k square) minus (k square) plus (k power 4) is less than or equal to r2, 3 is less than or equal to minus k square) plus square root of (1) minus (k square) minus (k square) plus (k power 4).
Implies, (minus k square) (minus 1) (minus k square) is less than or equal to r2, 3 is less than or equal to minus (k square) plus (1) minus (k square).
Therefore, (minus 1) less than or equal to (r2, 3) less than or equal to (1) minus 2 into (k square).

4. Result 4

Prove that (b1, 2 point 3) into (b2, 3 point 1) into (b3, 1 point 2) is equal to (r1, 2 point 3) into (r2, 3 point 1) into (r3, 1 point 2).
To prove the above results, let us consider the expressions of b in terms of r and sigma.
That is, (b1, 2 point 3) is equal to (r1, 2 point 3) into (sigma 1 point 3) divided by (sigma 2 point 3)
(b2, 3 point 1) is equal to (r2, 3 point 1) into (sigma 2 point 1) divided by (sigma 3 point 1), and (b3, 1 point 2) is equal to (r3, 1 point 2) into (sigma 3 point 2) divided by (sigma 1 point 2).
Now let us consider the left hand side of given expression and substitute for b's we get,
(b1, 2, point 3) into (b2, 3 point 1) into (b3, 1 point 2) is equal to (r1, 2 point 3) into (sigma 1 point 3) into (sigma 2 point 3), into (r2, 3 point 1) into (sigma 2 point 1) divided by (sigma 3 point 1), into (r3, 1 point 2) into (sigma 3 point 2) divided by (sigma 1 point 2).
On simplification we get,
(r1, 2 point 3) into (r2, 3 point 1) into (r3, 1 point 2).
Hence the proof.

5. Result 5

If (r1, 2) is equal to (r2, 3) is equal to (r 3, 1) is equal to row, which is not equal to one, then (r1, 2 point 3) is equal to (r2, 3 point 1) is equal to (r3, 1 point 2) is equal to (row) divided by (1 plus row). And (R1 point 2, 3) is equal to (R2 point 1, 3) is equal to (R3 point 1, 2) is equal to (row) into (square root of 2) divided by (square root of 1) plus (row).
Proof:
We know that (r 1, 2 point 3) is equal to (r1, 2) minus (r1, 3) into (r2, 3) divided by (square root of 1) minus (r1, 3 square) into (1 minus r2, 3 square).
Substituting (r1, 2) is equal to (r1, 3) is equal to (r2, 3) is equal to (row), we get,
(r1, 2 point 3) is equal to (row) minus (row square) divided by (square root of 1) minus (row square) into (1 minus row square).
Is equal to (row) into (1 minus row) divided by (1 minus row square)
is equal to (row) divided by (1 plus row).

Similarly, r2, 3 point 1 is equal to r2, 3 minus r2, 1 into r3, 1 divided by square root of 1 minus r2, 1 square into 1 minus r3, 1 square
is equal to row minus row square divided by square root of 1 minus row square into 1 minus row square
is equal to row by 1 plus row.
Finally row 3, 1 point 2 is equal to r3, 1 minus r3, 2 into r1, 2 divided by Square root minus r3, 2 square into 1 minus r1, 2 square
is equal to row minus row square divided by square root of 1 minus row square into 1 minus row square
is equal to row by 1 plus row.
Observe from above three expressions that,
r1, 2 point 3 is equal to r2, 3 point 1 is equal to r3, 1 point 2 is equal to row divided by 1 plus row.

Now let us consider the multiple correlation coefficient.
R1 point 2, 3 square is equal to r1, 2 square plus r1, 3 square minus 2 into r1, 2 into r1, 3 into r2, 3 divided by 1 minus r2, 3 square.
On substitution, we get,
R1 point 2, 3 square is equal to row square plus row square minus 2 into row cube divided by 1 minus row square
Is equal to 2 into row square into 1 minus row divided by 1 plus row into 1 minus row is equal to 2 into row square dived by 1 plus row.

Similarly consider
R2 point 1, 3 square is equal to r2, 1 square plus r2, 3 square minus 2 into r2, 1 into r2, 3 into r1, 3 divided by 1 minus r1, 3 square
Is equal to row square plus row square minus 2 into row cube divided by 1 minus row square
Is equal to 2 into row square dived by 1 plus row.
And finally R3 point 1, 2 square is equal to r3, 1 square plus r3, 2 square minus 2 into r3, 1 into r3, 2 into r1, 2 divided by 1 minus r1, 2 square
Is equal to row square plus row square minus 2 into row cube divided by 1 minus row square
Is equal to 2 into row square dived by 1 plus row.

Observe that we got,
R1 point 2, 3 square is equal to R2 point 1, 3 square is equal to R3 point 1, 2 is equal to 2 into row square dived by 1 plus row.

By taking square root, we get,
R1 point 2, 3 is equal to R2 point 1, 3 is equal to R3 point 1, 2 is equal to row into square root of 2 dived by square root of 1 plus row.
Hence the result.

Here's a summary of our learning in this session:
- Related results of multiple and partial correlation and regression