Frequently Asked Questions

1. What do you mean by multiple correlation coefficient.

Answer:
In tri-variate distribution in which each of the variables X_1, X_2, and X_3 has N observations, the multiple correlation coefficient between X_1 on X_2 and X_3 is, usually denoted by $R_{1.23}$ is the simple correlation coefficient between X_1 and the joint effect of X_2 and X_3 on X_1. In other words $R_{1.23}$ is the correlation coefficient between X_1 and its estimated value as given by the plane of regression of X_1 on X_2 and X_3.

2. What do you mean by Partial Regression

Answer:
The regression between only two variates eliminating the linear effect of other variates in them is called the partial regression.

3. Give an expression for multiple correlation coefficient in terms of total correlation coefficients.

Answer:

$$R_{1.23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}$$

4. Show that the correlation coefficient between the residuals $X_{1.23}$ and $X_{2.13}$ is equal and opposite to that between $X_{1.3}$ and X_{23}

Answer:
The correlation coefficient between $X_{1.23}$ and $X_{2.13}$ is given by

$$\frac{Cov(X_{1.23}, X_{2.13})}{\sigma_{1.23}\sigma_{2.13}} = \frac{\Sigma X_{1.23} X_{2.13}}{N\sigma_{1.23}\sigma_{2.13}} = \frac{\frac{1}{N}\Sigma X_{2.13}(X_1 - b_{12.3}X_2 - b_{13.2}X_3)}{\sigma_{1.23}\sigma_{2.13}}$$

$$= -b_{12.3}\frac{\Sigma X_{2.13}X_2}{N\sigma_{1.23}\sigma_{2.13}} = -b_{12.3}\frac{\Sigma X_{2.13}^2}{N\sigma_{1.23}.\sigma_{2.13}} = -b_{12.3}\frac{\sigma_{2.13}}{N\sigma_{1.23}} = -b_{12.3}\frac{\sigma_2(\sqrt{\omega/\omega_{22}})}{\sigma_1(\sqrt{\omega/\omega_{11}})}$$

where

$$\omega = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}, \quad \omega_{11} = \begin{vmatrix} 1 & r_{23} \\ r_{32} & 1 \end{vmatrix} = 1 - r_{23}^2, \quad \omega_{22} = \begin{vmatrix} 1 & r_{13} \\ r_{31} & 1 \end{vmatrix} = 1 - r_{31}^2$$

$$\therefore r(X_{1.23}X_{2.13}) = -b_{12.3}\frac{\sigma_2}{\sigma_1}\sqrt{\frac{1-r_{23}^2}{1-r_{13}^2}} = -b_{12.3}\frac{\sigma_{2.3}}{\sigma_{1.3}}$$ [Since $\sigma_{2.3}^2 = \sigma_2^2(1-r_{23}^2)$ and $\sigma_{1.3}^2 = \sigma_1^2(1-r_{13}^2)$]

$$\therefore r(X_{1.23}X_{2.13}) = -\frac{Cov(X_{1.3}, X_{2.3})}{\sigma_{2.3}^2}\frac{\sigma_{2.3}}{\sigma_{1.3}} = -\frac{Cov(X_{1.3}, X_{2.3})}{\sigma_{2.3}\sigma_{1.3}} = r(X_{1.3}X_{2.3})$$

5. Show that if $X_3 = aX_1 + bX_2$, the three partial correlations are numerically equal to unity, $r_{13.2}$ having the sign of a, $r_{23.1}$, the sign of b and $r_{12.3}$, the opposite sign of a/b.

Answer:

Here we may regard X_3 as dependent on X_1 and X_2 which may be taken as independent variables. Since X_1 and X_2 are independent, they are uncorrelated.

$$\therefore r_{13}=\frac{Cov(X_1,X_3)}{\sqrt{V(X_1)V(X_3)}}=\frac{a\sigma_{1^2}}{\sqrt{\sigma_{1^2}(a^2\sigma_{1^2}+b^2\sigma_{2^2})}}=\frac{a\sigma_1}{k}$$, Where $k^2= a^2\sigma_1^2+ b^2\sigma_2^2$.

Similarly we get,

$$r_{23}=\frac{Cov(X_2,X_3)}{\sqrt{V(X_2)V(X_3)}}=\frac{b\sigma_{2^2}}{\sqrt{\sigma_{2^2}(a^2\sigma_{1^2}+b^2\sigma_{2^2})}}=\frac{b\sigma_2}{k}$$

Hence

$$r_{13.2}=\frac{(r_{23}-r_{21}r_{31})}{\sqrt{(1-r_{21^2})(1-r_{31^2})}}=\frac{a\sigma_1}{k}\cdot\frac{k}{\sqrt{k^2-b^2\sigma_{2^2}}}=\frac{a\sigma_1}{\sqrt{a^2\sigma_{1^2}}}=\frac{a\sigma_1}{|a|\sigma_1}=\pm 1$$

according as ‘a’ is positive or negative. Hence $r_{13.2}$ has the same sign as ‘a’. Again

$$r_{23.1}=\frac{(r_{23}-r_{21}r_{31})}{\sqrt{(1-r_{21^2})(1-r_{31^2})}}=\frac{b\sigma_2}{k}\cdot\frac{k}{\sqrt{k^2-a^2\sigma_{1^2}}}=\frac{b\sigma_2}{\sqrt{b^2\sigma_{2^2}}}=\frac{b\sigma_2}{|b|\sigma_2}=\pm 1$$

according as ‘b’ is positive or negative. Hence $r_{23.1}$ has the same sign as ‘b’.

Now,

$$r_{12.3}=\frac{(r_{12}-r_{13}r_{23})}{\sqrt{(1-r_{13^2})(1-r_{23^2})}}=-\frac{a\sigma_1}{k}\frac{b\sigma_2}{k}\cdot\frac{k^2}{\sqrt{(k^2-a^2\sigma_{1^2})(k^2-b^2\sigma_{2^2})}}$$

$$=-\frac{a\sigma_1 b\sigma_2}{\sqrt{a^2\sigma_{1^2}b^2\sigma_{2^2}}}=-\frac{ab}{\sqrt{a^2b^2}}=-\frac{a/b}{\sqrt{a^2/b^2}}=-\frac{a/b}{\pm|a/b|}=\mp 1$$

according as a/b is positive or negative. Hence $r_{12.3}$ has the sign opposite to that of a/b.

6. If r_{12} and r_{13} are given, show that r_{23} must lie in the range $r_{12}\, r_{13} \pm (1 - r_{12}^2 - r_{13}^2 + r_{12}^2\, r_{13}^2)^{½}$.
If $r_{12} = k$, and $r_{13} = -k$, show that r_{23} will lie between -1 and 1 - 2k square.

Answer:

$$r_{12.3^2}=\left[\frac{(r_{12}-r_{13}r_{23})}{\sqrt{(1-r_{13^2})(1-r_{23^2})}}\right]^2\le 1$$

Therefore $(r_{12} - r_{13}r_{23})^2 \leq (1 - r_{13}^2)(1 - r_{23}^2)$.

Implies, $r_{12}^2 + r_{13}^2 r_{23}^2 - 2\, 2r_{12} r_{13}r_{23} \leq 1 - r_{13}^2 - r_{23}^2 + r_{13}^2 r_{23}^2$

Implies, $r_{12}^2 + r_{13}^2 + r_{23}^2 - 2r_{12} r_{13}r_{23} \leq 1$

This condition holds for consistent values of r_{12}, r_{13} and r_{23}. And hence the above inequality can be rewritten as,

$r_{23}^2 - (2r_{12} r_{13})r_{23} + (r_{12}^2 + r_{13}^2 - 1) \leq 1$

Hence, for given values of r_{12} and r_{13}, r_{23} must lie between the roots of the quadratic (in r_{23}) equation,

$r_{23}^2 - (2r_{12} r_{13})r_{23} + (r_{12}^2 + r_{13}^2 - 1) = 0$

r2, 3 square minus 2 into r1, 2 into r1, 3 into r2, 3 plus r1, 2 square plus r1, 3 square minus 1 is equal to zero.

Which are given by,

$$r_{23} = r_{12} r_{13} \pm \sqrt{r_{12}^2 r_{13}^2 - (r_{12}^2 + r_{13}^2 - 1)}$$

Hence $r_{12} r_{13} - \sqrt{r_{12}^2 r_{13}^2 - (r_{12}^2 + r_{13}^2 - 1)} \leq r_{23} \leq r_{12} r_{13} + \sqrt{r_{12}^2 r_{13}^2 - (r_{12}^2 + r_{13}^2 - 1)}$ --------(*)

In other words, r_{23} must lie in the range, $r_{12} r_{13} \pm \sqrt{1 - r_{12}^2 - r_{13}^2 + r_{12}^2 r_{13}^2}$

In particular, if $r_{12} = k$ and $r_{13} = -k$, we get from (*)

$$-k^2 - \sqrt{1 - k^2 - k^2 + k^4}) \leq r_{23} \leq -k^2 + \sqrt{1 - k^2 - k^2 + k^4})$$

Implies, $-k^2 - (1 - k^2) \leq r_{23} \leq -k^2 + (1 - k^2)$.

Therefore $-1 \leq r_{23} \leq 1-2k^2$

7. Prove that $b_{12.3}\, b_{23.1}\, b_{31.2} = r_{12.3}\, r_{23.1}\, r_{31.2}$.

Answer:

$b_{12.3} = r_{12.3}\, \sigma_{1.3}/\sigma_{2.3}$, $b_{23.1} = r_{23.1}\, \sigma_{2.1}/\sigma_{3.1}$, $b_{31.2} = r_{31.2}\, \sigma_{3.2}/\sigma_{1.2}$

Now let us consider the left hand side of given expression and substitute for b's we get

$b_{12.3}\, b_{23.1}\, b_{31.2} = (r_{12.3}\, \sigma_{1.3}/\sigma_{2.3})(r_{23.1}\, \sigma_{2.1}/\sigma_{3.1})(r_{31.2}\, \sigma_{3.2}/\sigma_{1.2}) = r_{12.3}\, r_{23.1}\, r_{31.2}$

8. If $r_{12} = r_{23} = r_{31} = \rho \neq 1$, then $r_{12.3} = r_{23.1} = r_{31.2} = \rho/(1+\rho)$.

Answer:

We know that $$r_{12.3} = \frac{(r_{12} - r_{13} r_{23})}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}} = \frac{\rho - \rho^2}{\sqrt{(1-\rho^2)(1-\rho^2)}} = \frac{\rho(1-\rho)}{(1-\rho^2)} = \frac{\rho}{1+\rho}$$

Similarly, $r_{13.2}=\frac{(r_{13}-r_{12}r_{32})}{\sqrt{(1-r_{12}^2)(1-r_{32}^2)}}=\frac{\rho-\rho^2}{\sqrt{(1-\rho^2)(1-\rho^2)}}=\frac{\rho}{1+\rho}$

Finally, $r_{31.2}=\frac{(r_{31}-r_{32}r_{12})}{\sqrt{(1-r_{32}^2)(1-r_{12}^2)}}=\frac{\rho-\rho^2}{\sqrt{(1-\rho^2)(1-\rho^2)}}=\frac{\rho}{1+\rho}$

Observe from above three expressions that,

$r_{12.3} = r_{23.1} = r_{31.2} = \rho/(1+\rho)$

9. If $r_{12} = r_{23} = r_{31} = \rho \neq 1$, then $R_{1.23} = R_{2.13} = R_{3.12} = \rho\sqrt{2}/\sqrt{(1+\rho)}$.

Answer:

$$R_{1.23}^2=\frac{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}}{1-r_{23}^2}$$

$$R_{1.23}^2=\frac{\rho^2+\rho^2-2\rho^3}{1-\rho^2}=\frac{2\rho^2(1-\rho)}{1-\rho^2}=\frac{2\rho^2}{1+\rho}$$

Similarly, $R_{2.13}^2=\frac{r_{21}^2+r_{23}^2-2r_{21}r_{23}r_{13}}{1-r_{13}^2}=\frac{\rho^2+\rho^2-2\rho^3}{1-\rho^2}=\frac{2\rho^2}{1+\rho}$

$$R_{3.12}^2=\frac{r_{31}^2+r_{32}^2-2r_{31}r_{32}r_{12}}{1-r_{12}^2}=\frac{\rho^2+\rho^2-2\rho^3}{1-\rho^2}=\frac{2\rho^2}{1+\rho}$$

Observe that,

$$R_{1.23}^2=R_{2.13}^2=R_{3.12}^2=\frac{2\rho^2}{1+\rho}$$

By taking square root, we get, $R_{1.23} = R_{2.13} = R_{3.12} = \rho\sqrt{2}/\sqrt{(1+\rho)}$

10. With usual notations show that, $R_{1.23}^2 \geq r_{12}^2$

Answer:

We know that, $1-R_{1.23}^2=(1-r_{12}^2)(1-r_{13.2}^2)$.

If we ignore the second term,

$1-R_{1.23}^2 \leq 1-r_{12}^2$, since the second term is less than 1.

Implies, $R_{1.23}^2 \geq r_{12}^2$

11. Establish the equation of plane of regression for variates X_1, X_2, X_3 in the determinant form:

$$\begin{vmatrix} X_1/\sigma_1 & X_2/\sigma_2 & X_3/\sigma_3 \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix} = 0$$

Answer:

We have given the above determinant is equal to zero.

ie. $\frac{X_1}{\sigma_1}\omega_{11}+\frac{X_2}{\sigma_2}\omega_{12}+\frac{X_2}{\sigma_3}\omega_{13}=0$

12. Show that if $r_{12}=r_{13}=0$, then $R_{1.23}=0$. What is the significance of this result in regard to the multiple regression equation of X_1 on X_2 and X_3?

Answer:

We know that, $$R_{1.23}^2=\frac{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}}{1-r_{23}^2}$$

If , $r_{12}=r_{13}=0$, then $R_{1.23}^2 = 0$. Implies, $R_{1.23} = 0$

That means X_1 is uncorrelated with any of other variables.

13. For what value of $R_{1.23}$ will X_2 and X_3 be uncorrelated with X_1?

Answer:

When $R_{1.23} = 0$, X_2 and X_3 are uncorrelated with X_1.

14. Given the data $r_{12}=0.6$, $r_{13}=0.4$, find the value of r_{23} so that $R_{1.23}$ is unity.

Answer

$R_{1.23}=1$ implies, $R_{1.23}^2=1$. We know that,

$$R_{1.23}^2=\frac{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}}{1-r_{23}^2}$$

By substitution, we get,

$1 - r_{23}^2=0.52-0.48r_{23}$

ie., $r_{23}^2-0.48r_{23}-0.48=0$

On solving the above quadratic equation, we get,

$r_{23}=-0.986$ or $r_{23}=1.946$

Since correlation coefficient cannot greater than 1, $r_{23}=-0.986$.

15. Prove that $R_{12.3}^2=b_{12.3}r_{12}\frac{\sigma_2}{\sigma_1}+b_{13.2}r_{13}\frac{\sigma_3}{\sigma_1}$

Answer:

Here we consider the right hand side of the above expression and try to get left hand side.

We know that,

$$b_{12.3}=r_{12.3}\frac{\sigma_1\sqrt{1-r_{13}^2}}{\sigma_2\sqrt{1-r_{23}^2}} \text{ and } b_{13.2}=r_{13.2}\frac{\sigma_1\sqrt{1-r_{12}^2}}{\sigma_3\sqrt{1-r_{32}^2}}$$

RHS, $b_{12.3}r_{12}\frac{\sigma_2}{\sigma_1}+b_{13.2}r_{13}\frac{\sigma_3}{\sigma_1}=r_{12.3}\frac{\sigma_1\sqrt{1-r_{13}^2}}{\sigma_2\sqrt{1-r_{23}^2}}r_{12}\frac{\sigma_2}{\sigma_1}+r_{13.2}\frac{\sigma_1\sqrt{1-r_{12}^2}}{\sigma_3\sqrt{1-r_{32}^2}}r_{13}\frac{\sigma_3}{\sigma_1}$

$$=r_{12.3}\frac{\sqrt{1-r_{13}^2}}{\sqrt{1-r_{23}^2}}r_{12}+r_{13.2}\frac{\sqrt{1-r_{12}^2}}{\sqrt{1-r_{32}^2}}r_{13}$$

Substituting for $r_{12.3}$ and $r_{13.2}$, we get,

$$=\frac{r_{12}-r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}\frac{\sqrt{1-r_{13}^2}}{\sqrt{1-r_{23}^2}}r_{12}+\frac{r_{13}-r_{12}r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{32}^2)}}\frac{\sqrt{1-r_{12}^2}}{\sqrt{1-r_{32}^2}}r_{13}$$

$$=\frac{r_{12}^2-r_{12}r_{13}r_{23}}{(1-r_{23}^2)}+\frac{r_{13}^2-r_{12}r_{13}r_{32}}{(1-r_{23}^2)}=\frac{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}}{(1-r_{23}^2)}=R_{1.23}^2$$

Hence the proof.