1. Introduction

Welcome to the series of e-learning modules on Multiple Correlation and Regression in Three Variables. Here we shall discuss about the multiple correlation, its coefficient & properties, and expression for multiple regression in terms of total and partial correlation coefficient.

By the end of this session, you will be able to explain:

- Multiple correlation
- Multiple regression
- Properties of multiple regression
- Expression for multiple regression in terms of total and partial correlation coefficients

When the values of one variable are associated with or influenced by other variable Karl Pearson's coefficient of correlation can be used as a measure of linear relationship between them.
Say, for example:
The age of husband and wife,
the height of father and son,
the supply and demand of a commodity and so on,

But sometimes there is interrelation between many variables, and the value of one variable may be influenced by many others.
An Example would be:
The yield of crop per acre, say 'X_1' depends upon quality of seed say 'X_2', fertility of soil, say 'X_3' , fertilizer used say 'X_4', irrigation facilities say X_5, weather conditions say 'X_6' and so on.
Whenever we are interested in studying the joint effect of a group of variables upon a variable not included in that group, our study is that of multiple correlations and multiple regressions.

2. Coefficient of Multiple Correlation

Now let us discuss about the coefficient of multiple correlation.
In a tri-variate distribution in which each of the variables ‘X1’, ‘X2’, and ‘X3’ has ‘N’ observations, the multiple correlation coefficient between ‘X1’ on ‘X2’ and ‘X3’ is, usually denoted by (R ‘1 point 2-3’), which is the simple correlation coefficient between ‘X1’ and the joint effect of X2 and ‘X3’ on ‘X1’.

In other words (R ‘1 point 2-3’) is the correlation coefficient between ‘X1’ and its estimated value as given by the plane of regression of ‘X1’ on ‘X2’ and ‘X3’ namely,
(e ‘1 point 2-3’) is equal to [(b ‘1-2 point 3’) into ‘X2’] plus [(b ‘1-3 point 2’) into ‘X3’].
We have (X ‘1 point 2-3’) is equal to ‘X1’ minus (b ‘1-2 point 3’) into ‘X2’ minus (b ‘1-3 point 2’) into ‘X3’ is equal to ‘X1’ minus (e ‘1 point 2-3’)
Implies, (e ‘1 point 2-3’) is equal to ‘X1’ minus (X ‘1 point 2-3’).

Since X-i’s are measured from their respective means, we have:
[Expectation of (X ‘1 point 2-3’)] is equal to ‘zero’ and [expectation of (e ‘1 point 2-3’)] is equal to zero.
Since Expectation of ‘X-I’ is equal to ‘zero’ for ‘i’ is equal to 1, 2 and 3.
By definition,
(R ‘1 point 2-3’) is equal to covariance of ‘X1’, (e ‘1 point 2-3’) divided by (square root of variance of ‘X1’) into (variance of (e ‘1 point 2-3’)).
Covariance of ‘X1’, (e ‘1 point 2-3’) is equal to (expectation of ‘X1’) minus (E of ‘X1’) into (e ‘1 point 2-3’) minus [E of (e ‘1 point 2-3’].
Is equal to (expectation of ‘X1’) into (e ‘1 point 2-3’)
Is equal to ‘1 by N’ into (summation ‘X1’) into (e ‘1 point 2-3’).
Is equal to ‘1 by N’ into (summation ‘X1’) into [X1 minus (X ‘1 point 2-3’)].

Is equal to ‘1 by N’ into (summation ‘X1’ square) minus [‘1 by N’ into ‘summation X1’ into (X ‘1 point 2-3’)].
Is equal to ‘1 by N’ into (summation ‘X1’ square) minus ‘1 by N’ into [summation (X ‘1 point 2-3’) square].
Is equal to ‘sigma 1 square’ minus (sigma ‘1 point 2-3’ square).
Also [variance of (e ‘1 point 2-3’)] is equal to [expectation of (e ‘1 point 2-3’) square].
Is equal to ‘I by N’ into summation [(e ‘1 point 2-3’) square]
Is equal to ‘1 by N’ into summation ‘X1’ minus (X ‘1 point 2-3’) square.
Is equal to ‘1 by N’ into (summation ‘X1’ square) plus [(X ‘1 point 2-3’) square] minus [2 into ‘X1’ into (X ‘1 point 2-3’)].

Is equal to ‘1 by N’ into (summation ‘X1’ square) plus ‘1 by N’ into [summation (X ‘1 point 2-3’) square minus [‘2 by N’ into ‘summation X1’ into (X ‘1 point 2-3’)]

Is equal to ‘1 by N’ into (summation ‘X1’ square) plus [‘1 by N’ into summation (X ‘1 point 2-3’) square] minus [‘2 by N’ into summation (X ‘1 point 2-3’ square)].
Is equal to (sigma 1 square) minus (sigma ‘1 point 2-3’ square).

Substituting in (R ‘1 point 2-3’), we get,
(R ‘1 point 2-3’) is equal to (sigma 1 square) minus (sigma ‘1 point 2-3’ square) divided by [square root of (sigma 1 square)] into [(sigma 1 square) minus (sigma ‘1 point 2-3’ square)].
Implies, (R ‘1 point 2-3’ square) is equal to (sigma 1 square) minus (sigma ‘1 point 2-3’ square) divided by (sigma 1 square) is equal to [‘1’ minus (sigma ‘1 point 2-3’ square) divided by (sigma 1 square)
Implies [‘1’ minus (R ‘1 point 2-3’ square) is equal to (sigma ‘1 point 2-3’ square) divided by (sigma 1 square).

The above expression we can write in terms of omega,
(‘1’ minus (R ‘1 point 2-3’) square) is equal to (‘omega’ divided by ‘omega 1-1’)
Where omega is equal to (determinant of ‘1’, ‘r 1-2’, ‘r 1-3’, ‘r 2-1’, ‘1’, ‘r 2-3’, ‘r 3-1’, ‘r-3-2’, ‘1’) is equal to ['1’ minus (‘r 1-2’ square) minus (‘r 1-3’ square) minus (r ‘2-3 square’)] plus (‘2’ into ‘r 1-2’ into ‘r 1-3’ into ‘r 2-3’)
And (omega 1-1) is equal to (determinant of ‘1’, ‘r 2-3’, ‘r 3-2’, ‘1’
is equal to (‘1’ minus ‘r 2-3’ square).

Hence we get, (R ‘1 point 2-3’ square) is equal to (‘1’ minus ‘omega’) divided by (omega 1-1) is equal to (‘r 1-2’ square) plus (‘r 1-3’ square) minus (‘2’ into ‘r 1-2’ into ‘r 1-3’ into ‘r 2-3’) divided by (‘1’ minus ‘r 2-3’ square).
This formula expresses the multiple correlation coefficient in terms of the total correlation coefficient between the pairs of variables.

3. Remarks of Multiple Correlation

Now consider the following remarks.
(R ‘1 point 2-3’ square) is ‘less than or equal’ to ‘1’
implies, ‘1’ minus (R ‘1 point 2-3’ square) is equal to (‘omega’ by ‘omega 1-1’) is ‘greater than or equal to’ ‘zero’.
But, ‘omega 1-1’ is equal to (‘1’ minus ‘r 2-3’ square) greater than ‘zero’,
Since, (modulus of ‘r 2-3’) is ‘less than or equal to’ ‘1’.
Therefore, ‘omega’ is ‘greater than or equal to’ ‘zero’.
Also ‘omega 2-2’ is equal to (‘minus 1’ power ‘2 plus 2’) into (‘1’ minus ‘r 1-3’ square) is ‘greater than or equal to’ ‘zero’, and
‘omega 3-3’ is equal to (‘minus 1’ power ‘3 plus 3’) into (‘1’ minus ‘r 1-2’ square) is ‘greater than or equal to’ ‘zero’.
In general, ‘omega-i’ is ‘greater than or equal to’ ‘zero’ for (‘i’ is equal to 1, 2 and 3).

The multiple correlation coefficients (R ‘2 point 1-3’ square) and (R ‘3 point 1-2’ square) are given by:
(R ‘2 point 1-3’ square) is equal to (‘r 2-1’ square) plus (‘r 2-3’ square) minus (‘2’ into ‘r 2-1’ into ‘r 2-3’ into ‘r 1-3’) divided by (‘1’ minus ‘r 1-3’ square) and
(R ‘3 point 1-2’ square) is equal to (‘r 1-3’ square) plus (‘r 1-2’ square) minus (‘2’ into ‘r 1-2’ into ‘r 1-3’ into ‘r 2-3’) divided by (1 minus ‘r 1-2’ square).

In the case of ‘n’ variables ‘X1’, ‘X2’, etc ‘X-n’, the multiple correlation coefficients of ‘X1’ on ‘X2’, ‘X3’ etc ‘X-n’ is usually denoted by (R ‘1 point 2-3’ etc n), is the correlation coefficient between ‘X1’ and (e ‘1 point 2-3’ etc n) is equal to ‘X1’ minus (X ‘1 point 2-3’ etc. n)
Therefore (R ‘1 point 2-3’ etc n) is equal to covariance of ‘X1’, (e ‘1 point 2-3’ etc n) divided by [square root of ‘variance of X1’ into ‘Variance of (e ‘1 point 2-3’ etc n)].
Covariance of ‘X1’, (e ‘1 point 2-3’ etc n) is equal to ‘1 by N’ summation ‘X1’ into (e ‘1 point 2-3’ etc n)
Is equal to ‘1 by N’ into summation ‘X1’ into (‘X1’ minus (X ‘1 point 2-3’ etc n)

Is equal to ‘1 by N’ into (summation ‘X1’ square) minus ‘1 by N’ into summation ‘X1’ into (X ‘1 point 2-3’ etc n)
Is equal to ‘1 by N’ into (summation ‘X1’ square) minus ‘1 by N’ summation (X ‘1 point 2-3’ etc n square)
is equal to (sigma 1 square) minus (sigma ‘1 point 2-3’ etc n square).
Also, Variance of (e ‘1 point 2-3’ etc n) is equal to ‘1 by N’ summation (e ‘1 point 2-3’ etc n square)
Is equal to ‘1 by N’ [summation ‘X1’ minus (X ‘1 point 2-3’ etc n whole square)].
Is equal to ‘1 by N’ into (summation ‘X1’ square) plus (X ‘1 point 2-3’ etc n square) minus [‘2’ into ‘X1’ into (X ‘1 point 2-3’ etc n)].

Is equal to ‘1 by N’ into (summation ‘X1’ square) plus ‘1 by N’ summation (X ‘1 point 2-3’ etc n square) minus [‘2’ into ‘1 by N’ summation ‘X1’ into (X ‘1 point 2-3’ etc n)].
Is equal to ‘1 by N’ into (summation ‘X1’ square) plus ‘1 by N’ summation (X ‘1 point 2-3’ etc n square) minus [‘2 by N’ summation (X ‘1 point 2-3’ etc n) square].
Is equal to (sigma 1 square) minus (sigma ‘1 point 2-3’ etc., n square).
Therefore, (R ‘1 point 2-3’ etc., n) is equal to (sigma 1 square) minus (sigma ‘1 point 2-3’ etc n square) divided by square-root of (sigma 1 square) into (sigma 1 square) minus (sigma ‘1 point 2-3’ etc n square).
Is equal to (sigma 1 square) minus (sigma ‘1 point 2-3’ etc n) divided by (sigma 1 square) whole power ‘half’.
Implies (R ‘1 point 2-3’ etc., n square) is equal to [‘1’ minus (sigma ‘1 point 2-3’ etc n square) divided by (sigma 1 square)
Is equal to ‘1’ minus ‘omega’ divided by ‘omega 1-1’.

4. Multiple Correlation in terms of Total and Partial Correlation Coefficients

Now let us obtain the expression for multiple correlation in terms of total and partial correlation coefficients.
Show that 1 minus (R ‘1 point 2-3’ square) is equal to (‘1’ minus ‘r 1-2’ square) into (‘1’ minus ‘r 1-3 point 2’ square).
Deduce that,
(R ‘1 point 2-3’) is greater than or equal to ‘r 1-2’
(R ‘1 point 2-3’ square) is equal to (‘r 1-2’ square) plus (‘r 1-3’ square) if ‘r 2-3’ is equal to ‘zero’.
1 minus (R ‘1 point 2-3’ square) is equal to (‘1’ minus ‘row’) into (‘1’ plus ‘2 into row’) divided by (‘1’ plus ‘row’), provided all coefficients of zero order are equal to ‘row’.
If (R ‘1 point 2-3’) is equal to ‘zero’, ‘X1’ is uncorrelated with any other variables, that is ‘r 1-2’ is equal to ‘r 1-3’ is equal to ‘zero’.

We know that [‘1’ minus (R ‘1 point 2-3’) square] is equal to (‘1’ minus ‘r 1-2’ square) plus (‘r 1-3’ square) minus (‘2’ into ‘r 1-2’ into ‘r 1-3’ into ‘r 2-3’) divided by (‘1’ minus ‘r 2-3’ square).
Is equal to ‘1’ minus (‘r 2-3’ square) minus (‘r 1-2’ square) minus (‘r 1-3’ square) plus (‘2’ into ‘r 1-2’ into ‘r 1-3’ into ‘r 2-3’) divided by (‘1’ minus ‘r 2-3’ square).

Also, (‘1’ minus ‘r 1-3 point 2’ square)
is equal to ‘1’ minus (‘r 1-3 square’) minus (‘r 1-2’ into ‘r 2-3’ whole square) divided by (‘1’ minus ‘r 1-2’ square) into (‘1’ minus ‘r 2-3’ square)
Is equal to ‘1’ minus (‘r 1-2’ square) minus (r ‘2-3’ square) minus (r ‘1-3’ square) plus (‘2’ into ‘r 1-2’ into ‘r 1-3’ into ‘r 2-3’) divided by [‘1’ minus (‘r 1-2’ square) into ‘1’ minus (‘r 2-3’ square)].
Implies, [‘1’ minus (R ‘1 point 2-3’) square] is equal to [‘1’ minus (‘r 1-2’ square) into ‘1’ minus (‘r 1-3 point 2’ square).

After showing the above expression let us deduce the different results as follows.
To prove the first one, let us start as follows.
Since (modulus of ‘r 1-3 point 2’ is ‘less than or equal to’ ‘1’), we have,
‘1’ minus (R ‘1 point 2-3’ square) is ‘less than or equal to’ (‘1 minus ‘r 1-2’ square) implies, (R ‘1 point 2-3’) is ‘greater than or equal to’ ‘r 1-2’.
To prove the second one, consider:
(‘r 1-3 point 2’ square) is equal to (‘r 1-3’ square) plus (‘r 1-2’ into ‘r 2-3’ whole square) divided by [square root of ‘1’ minus (‘r 1-2’ square) into ‘1’ minus (‘r 2-3’ square)]

Is equal to (‘r 1-3’ square) divided by (square root of ‘1’ minus ‘r 1-2’ square), if ‘r 2-3’ is equal to ‘zero’.
Therefore [‘1’ minus (R ‘1 point 2-3’ square)] is equal to (‘1’ minus ‘r 1-2’ square) into (‘1’ minus ‘r 1-3’ square) by (‘1’ minus ‘r 1-2’ square)
is equal to ‘1’ minus (‘r 1-2’ square) minus (‘r 1-3’ square)
Hence (R ‘1 point 2-3’ square) is equal to (‘r 1-2’ square) plus (‘r 1-3’ square) if ‘r 2-3’ is equal to ‘zero’.

To prove the third one, let us consider that the all coefficients of zero order are equal to row.
That is ‘r 1-2’ is equal to ‘r 1-3’ is equal to ‘r 2-3’ is equal to ‘row’.
Therefore (r ‘1-3 point 2’) is equal to (‘row’ minus ‘row-square’) divided by (square root of ‘1’ minus ‘row square’ into ‘1’ minus ‘row square’)
Is equal to (‘row’ into ‘1 minus row’) divided by (‘1’ minus ‘row square’)
is equal to ‘row’ divided by ‘1 plus row’.
Hence, by substituting in [1 minus (R ‘1 point 2-3’) square], which is given in terms of partial and total correlation coefficient, we get,
[1 minus (R ‘1 point 2-3’) square] is equal to [‘1’ minus ‘row square’ into (‘1’ minus ‘row square’ by ‘1 plus row’ the whole square]
is equal to ‘1 minus row’ into (‘1’ plus ‘2 into row’) divided by (‘1’ plus ‘row’).

To prove the last one, we again consider [1 minus (R ‘1 point 2-3’) square],
which is given in terms of partial and total correlation coefficient.
In this expression, if (R ‘1 point 2-3’) is equal to ‘zero’, we get,
‘1’ is equal to (‘1’ minus ‘r 1-2’ square) into (‘1’ minus ‘r 1-3 point 2’ square)
Since ‘zero’ ‘less than or equal’ to (‘r 1-2’ square) ‘less than or equal to’ ‘1’, and ‘zero’ ‘less than or equal to’ (‘r 1-3 point 2’ square) less than or equal to ‘1’, will hold if and only if (‘r 1-2’ is equal to ‘zero’) and (r ‘1-3 point 2’ is equal to ‘zero’).
Now, (r ‘1-3 point 2’) is equal to ‘zero’
implies, [(r ‘1-3’) minus (r ‘1-2’) into (r ‘3-2’)] divided by [(square root of ‘1’ minus ‘r 1-2’ square) into (‘1’ minus ‘r 3-2’ square)] is equal to ‘zero’
Implies, ‘r 1-3’ is equal to ‘zero’.
Thus if (R ‘1 point 2-3’) is equal to ‘zero’ then ‘r 1-3’ is equal to ‘r 1-2’ is equal to ‘zero’.
That is ‘X1’ is uncorrelated with ‘X2’ and ‘X3’.

5. Properties of Multiple Correlation Coefficient

1. Multiple correlation coefficient measures the closeness of the association between the observed values and the expected values of a variable obtained from the multiple linear regression of that variable on other variables.
2. Multiple correlation coefficient between observed values and expected values when the expected values are calculated form a linear relation of the variables determined by the method of least square, is always greater than that where expected values are calculated from any other linear combination of the variables.
3. Since (R '1 point 2-3') is the simple correlation between 'X1' and (e '1 point 2-3'), it must lie between 'minus 1' and 'plus 1'. But (R '1 point 2-3') is a non-negative quantity, 'zero' less than or equal to (R '1 point 2-3') less than or equal to '1'.
4. If (R '1 point 2-3') is equal to '1', then association is perfect and all the regression residuals are zero, and as such (Sigma '1 point 2-3' square) is equal to 'zero'. In this case, since 'X1' is equal to (e '1 point 2-3'), the predicted value of 'X1', the multiple linear regression equation of 'X1' on 'X2' and 'X3' may be said to be a perfect prediction formula.
5. If (R '1 point 2-3') is equal to 'zero', then all total and partial correlations involving 'X1' are 'zero'. So 'X1' is completely uncorrelated with all the other variables in this case and the multiple regression equation fails to throw any light on the value of 'X1' when 'X2' and 'X3' are known.
6. (R '1 point 2-3') is not less than any total correlation coefficient. ie., (R ' 1 point 2-3') greater than 'r 1-2' or 'r 1-2' or 'r 2-3'.

From the data relating to the yield of dry bark (X1), height (X2) and girth (X3) for 18 cinchona plants, the following correlation coefficients were obtained.
'r 1-2' is equal to '0.72'; 'r 1-3' is equal to '0.72' and 'r 2-3' is equal to '0.52'.
Find the multiple correlation coefficient.,
(R '1 point 2-3') square is equal to [('r 1-2' square) plus ('r 1-3' square) minus ('2' into 'r 1-2' into 'r 1-3' into 'r 2-3')] divided by ('1' minus 'r 2-3' square)
Is equal to [('0.77' square) plus ('0.72' square) minus ('2' into '0.77' into '0.72' into '0.52')] divided by ('1' minus '0.52' square) is equal to '0.7334'.
Therefore (R '1 point 2-3') is equal to (square root of '0.7334')
is equal to ('0.8564'), since multiple correlation coefficient is non-negative.

Here's a summary of our learning in this session:

- Multiple correlation
- Multiple regression
- Properties of multiple regression
- Expression for multiple regressions in terms of total and partial correlation coefficients.