1. Introduction

Welcome to the series of e-Learning modules on Poisson Distribution. In this module we are going to cover the definition of Poisson distribution, moments, mgf, cgf, skewness, kurtosis, recurrence relation for probabilities, mode, fitting Poisson distribution to the given data and its applications.

By the end of this session, you will be able to explain:

- Poisson Distribution
- Applications
- Moments and Moment Generating Function
- Nature of the distribution
- Cumulants Generating Function
- Fitting of Poisson distribution

Introduction

Poisson distribution is discovered by the French Mathematician and physicist Simeon Denis Poisson (1781-1840) who published it in 1837.

Poisson distribution is a limiting case of binomial distribution under the following conditions:

(i) n, the number of trials is indefinitely large, that is n tends to infinity

(ii) p, the constant probability of success for each trial is indefinitely small, that is p tends to zero and

(iii) np = lambda, (say), is finite. Thus, p is equal to lambda by n and q= 1 minus lambda by n, where lambda is a positive real number

The probability function of Poisson distribution is found as limiting case of Binomial distribution.

The probability of x successes in a series of n independent trials is:

P of x is equal to n c x, p power x, q power n – x, x takes value 0, 1, 2, etc..n

We want the limiting form of p of x under the above condition. Hence by substituting p in terms of lambda,

Limit n tends to infinity, p of x is equal to limit n tends to infinity n factorial by x factorial into n minus x factorial into lambda by n power x into 1 minus lambda by n whole power n minus x.

Using Stirling’s approximation for n factorial as n tends to infinity, namely,

Limit n tends to infinity; n factorial is equal to root 2 pi into e power minus n into n to the power n plus half, we get,

Limit n tends to infinity p of x is equal to limit n tends to infinity, root 2 pi into e power minus n into n to the power n plus half divided by x factorial into root 2 pi into e power minus of n-x into n to the power n minus x plus half into lambda by n power x, into 1 minus lambda by n whole power n-x.

By cancelling e power minus n and root 2 pi in both numerator and denominator and also taking e lambda power x, e power x and x factorial outside the limit, and combining the powers of n we get,

Lambda power x by e power x into x factorial, into limit n tends to infinity, n power n minus x plus half divided by n minus x power n minus x plus half into 1 minus lambda by n whole to the power n minus x

By taking n power n minus x plus half outside in the denominator and cancelling with the numerator, in the above expression we get

Lambda power x by e power x into x factorial, limit n tends to infinity 1 minus lambda by n whole power n minus x divided by 1 minus x by n whole power n minus x plus half

Now let us split the limits for each term in both numerator and denominator

Hence we get,

Lambda power x by e power x into x factorial, limit n tends to infinity 1 minus lambda by n whole power n into limit n tends to infinity 1 minus lambda by n whole power minus x divided by limit n tends to infinity 1 minus x by n whole power n into limit n tends to infinity, 1 minus x by n whole power minus x plus half.

In general we know that, limit n tends to infinity 1 minus a by n whole power n is equal to e power minus a.

Hence by applying this result in the above expression, we get,

Limit n tends to infinity p of x is equal to lambda power x by e power x into x factorial, into e power minus lambda into 1 divided by e power minus x into one as second terms in both numerator and denominator are independent of n in power.

Equals to e power minus lambda, lambda power x by x factorial, x takes value 0, 1, 2, etc., infinity.

2. Definition of Poisson Distribution

Hence we can define Poisson distribution as follows:
A random variable x is said to follow Poisson distribution with parameter lambda if its probability mass function is given by,
P of x is equal to e power minus lambda, lambda power x by x factorial for x is equal to 0, 1, 2, etc., infinity and zero otherwise.
And we write x follows Poisson lambda, read as x follows Poisson distribution with parameter lambda.

Following are some instances where Poisson distribution may be successfully employed:

- Number of deaths from a disease (not in the form of an epidemic) such as heart attack or cancer or due to snake bite.
- Number of suicides reported in a particular city.
- The number of defective material in a packing manufactured by a good concern.
- Number of faulty blades in a packet of 100.
- Number of air accidents in some unit of time.
- Number of printing mistakes at each page of the book.
- Number of telephone calls received at a particular telephone exchange in some unit of time or connections to wrong numbers in a telephone exchange.
- Number of cars passing a crossing per minute during the busy hours of a day.
- The number of fragments received by a surface area't' from a fragment atom bomb.
- The emission of radioactive (alpha) particles.

Poisson distribution has contributed a lot in the field of queuing theory.

Now let us find moments of the distribution.
First order moment mew one dash is equal to expectation of x, is equal to summation over x, x into p of x
Is equal to summation over x, x into e power minus lambda, lambda power x by x factorial.
Expanding x factorial as x into x minus 1 factorial and adding and subtracting 1 in the power of lambda, we get
Summation over x, x into e power minus lambda, lambda power x plus 1 minus one divided by x into x-1 factorial.
Is equal to lambda into e power minus lambda, summation over x from 1 to infinity, lambda power x – 1 divided by x minus 1 factorial.
Is equal to lambda into e power minus lambda into 1 plus lambda plus lambda square by 2 factorial plus lambda cube by 3 factorial etc.
Is equal to lambda into e power minus lambda into e power lambda
which is same as lambda
Hence, mean of the Poisson distribution is lambda.

3. Second ,Third and Fourth Raw Moment

Second raw moment is given by,
Mew 2 dash is equal to expectation of x square is equal to summation over x from zero to infinity x square into p of x.
Which is same as, summation over x, x into x minus 1 plus x into e power minus lambda, lambda power x by x factorial.
By splitting summation, and doing adjustments, we get
Summation over x, x into x – 1 into e power minus lambda, lambda power x by x factorial plus summation over x, x into e power minus lambda, lambda power x by x factorial.
Same as summation x into x-1 e power minus lambda, lambda power x minus two plus two by x into x -1 into x -2 factorial plus lambda
Is equal to e power minus lambda, lambda square into summation over x from 2 to infinity lambda power x – 2 divided by x -2 factorial plus lambda
Is equal to e power minus lambda, lambda square into e power lambda plus lambda
Is equal to lambda square plus lambda.

Third raw moment is given by, mew 3 dash is equal to expectation of x cube is equal to summation over x, x cube into p of x.
Is equal to summation over x, x into x – 1 into x – 2 plus 3 into x into x – 1 plus x into e power minus lambda, lambda power x by x factorial

Simplifying as we have done in earlier cases, we get
Mew 3 dash is equal to lambda cube plus 3 lambda square plus lambda.

Fourth raw moment is given by, mew 4 dash is equal to expectation of x power 4 is equal to summation over x, x power 4 into p of x.
Is equal to summation over x, x into x – 1 into x – 2 into x – 3 plus six x into x – 1 into x – 2 plus 7 into x into x – 1 plus x into e power minus lambda, lambda power x by x factorial.
On simplification we get,
Mew 4 dash is equal to lambda power four plus 6 lambda cube plus 7 lambda square plus lambda.

Having first four raw moments we can find second, third and fourth central moments as follows:

Mew 2 is equal to mew two dash minus mew one dash square is equal to lambda square plus lambda minus lambda square which is same as lambda.
Mew 3 is equal to mew 3 dash minus 3 into mew two dash into mew 1 dash plus 2 mew one dash cube.
Is equal to lambda cube plus 3 into lambda square plus lambda, minus 3 into lambda square plus lambda into lambda, plus 2 lambda cube.
On simplification, we get lambda.
Mew 4 is equal to mew 4 dash, minus 4 into mew 3 dash into mew 1 dash, plus 6 into mew 2 dash into mew 1 dash square, minus three into mew 1 dash power four

Is equal to lambda power 4 plus six lambda cube plus 7 into lambda square plus lambda, minus 4 into lambda cube plus 3 lambda square plus lambda into lambda, plus 6 into lambda square plus lambda into lambda square, minus 3 into lambda power 4

On simplification we get 3 lambda square plus lambda.

Observe that mean and variance of the Poisson distribution are equal to the parameter of the distribution λ.

4. Nature of Distribution

Now let us discuss the nature of the distribution
Coefficient of skewness is given by,
Beta 1 is equal to mew 3 square by mew 2 cube which is equal to lambda square by lambda cube, we get 1 by lambda.
Hence the Poisson distribution is always a skewed distribution.

Coefficient of Kurtosis is given by,
Beta 2 is equal to mew 4 by mew 2 square
is equal to 3 lambda square plus lambda divided by lambda square is equal to 3 plus 1 by lambda, which is greater than 3.
Hence Poisson distribution has leptokurtic curve.

Let us find mode of the distribution.
Mode is that value of x for which p(x) is maximum. Hence first we find the ratio of probabilities when X takes value x and x-1.
That is p of x by p of x – 1 is equal to e power minus lambda, lambda power x by x factorial, divided by e power minus lambda, lambda power x - 1by x – 1 factorial
which is equal to lambda by x.
Now let us consider the two cases such that lambda is an integer and not an integer.

Case – 1: when lambda is not an integer.
Let us suppose that lambda can be written as m + f, where m is the integral part of lambda and f is a fraction, which lies between zero and 1. Then we have,
P of 1 by p of zero is greater than 1, p of 2 by p of 1is greater than 1 etc., p of m by p of m – 1 is greater than 1.
And p of m+1 by p of m is less than 1, p of m+2 by p of m+1 is less than 1 etc..., p of n by p of n-1 is less than 1.

From above expressions, we can write, P of zero is less than p of 1 less than p of 2 less than etc., less than p of m-1 less than p of m greater than p of m+1 greater than p of m+2 etc., which shows that p of m is the maximum value. Hence in this case the distribution is unimodal and m, the integral part of lambda is the unique modal value.

Case – 2: when lambda is an integer.
Let us suppose that lambda can be written as m, an integer. Then we have
P of 1 by p of zero is greater than 1, p of 2 by p of 1is greater than 1 etc., p of m-1 by p of m – 2 is greater than 1.
P of m by p of m-1 is equal to 1.
And p of m+1 by p of m is less than 1, p of m+2 is less than p of m+1 etc.

From above expressions, we can write,
P of zero is less than p of 1 less than p of 2 less than etc., less than p of m-1 is equal to p of m greater than p of m+1 greater than p of m+2 etc.,
In this case we have two maximum values namely p of m – 1 and p of m and thus the distribution is bimodal and two modes are at m-1 and m. That is at lambda – 1 and lambda.

Now let us find the moment generating function.
M x of t is equal to expectation of e power tx
Is equal to summation over x, e power tx into p of x
Is equal to summation e power tx into e power minus lambda, lambda power x by x factorial
By combining the terms with power x, we get summation over x e power minus lambda into e power t into lambda whole power x by x factorial
Is equal to e power minus lambda, into e power lambda into e power t
Is equal to e power minus lambda into 1 minus e power t.

Cumulants Generating Function (cgf) is obtained by taking logarithm of mgf and is denoted by $K_X(t)$.

Having moment generating function we can write Cumulants Generating Function of Poisson distribution as follows:
K x of t is equal to logarithm of m x of t
Is equal to log e power minus lambda into 1 minus e power t
Is equal to minus lambda into 1 minus e power t
Using exponential expansion we get,
Minus lambda into 1 minus 1 plus t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 by 4 factorial etc.
Which is same as lambda into t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 by 4 factorial etc.
K_r is equal to r^{th} Cumulants are equal to the coefficient of t power r by r factorial in k x of t.
Observe that k r is equal to lambda for r is equal to 1, 2, 3, 4 etc
Hence for Poisson distribution, all the Cumulants are equal.

5. Sum of Independent Poisson Variates

Let us show that sum of independent Poisson variates is also a Poisson variate.
That is if x i, i = 1, 2, 3, etc., n are independent Poisson variates with parameters lambda i, i is equal to 1, 2, 3, etc., n respectively, then summation over i from 1 to n x i is also a Poisson variate with parameter summation lambda i.

We prove this property using moment generating function of the distribution.
We know that M x i of t is equal to e power minus lambda i into 1 minus e power t.
Consider M x1 plus x2 plus etc., plus x n of t.
Is equal to M x1 of t into M x2 of t into etc., into M xn of t, since x i's are independent.
Is equal to e power minus lambda 1 into 1 minus e power t, into e power minus lambda 2 into 1 minus e power t, into etc., into e power minus lambda n into 1 minus e power t.
Is equal to e power minus of lambda 1 plus lambda 2 plus etc., plus lambda n into 1 minus e power t.

Which is the mgf of Poisson distribution with parameter lambda 1 plus lambda 2 plus etc., plus lambda n. Hence by uniqueness theorem of mgf, summation x i is also a Poisson variate with parameter summation lambda i.

Result
If X and Y are independent Poisson variates, show that the conditional distribution of X given X + Y is binomial.
Proof
Let X and Y be independent Poisson variates with parameters lambda and mew respectively.
Then X + Y is also a Poisson variate with parameter lambda plus mew µ.

Consider conditional probability of x is equal to r given x + y = n
Equals to probability of x = r intersection x + y = n divided by probability of x + y = n
Is equal to probability of x = r intersection y = n –r divided by probability of x + y = n
Since x and y are independent, we get
probability of x = r into probability of y = n –r divided by probability of x + y = n
which is same as e power minus lambda, lambda power r by r factorial,
into e power minus mew, mew power n-r by n minus r factorial
divided by e power minus of lambda plus mew, lambda plus mew whole power n by n factorial

is equal to n factorial by r factorial into n-r factorial, into lambda by lambda plus mew whole power r, into mew by lambda plus mew whole power n-r
which is same as n c r, p power r into q power n minus r,
where p is equal to lambda by lambda plus mew and q =1-p
Hence the conditional distribution of X given X+Y = n is a binomial distribution with parameters n and p = lambda by lambda plus mew.

Let us discuss the method of Fitting a Poisson distribution to the given data.

First we find the recurrence relation for the probabilities of Poisson distribution.
Hence consider the ratio of probabilities when x takes value x and x-1 Which is equal to lambda by x, (is already done while finding mode of the distribution)
Therefore we get p of x is equal to lambda by x into p of x-1.

This formula provides a very convenient method of graduating the given data by a Poisson distribution. The only probability we need to calculate is p of zero which is given by p of zero equal to e power minus lambda, where lambda is estimated from the given data if it is not known using the estimator, sample mean. The other probabilities can be easily obtained by substituting x is equal to 1, 2, 3, etc.

That is p of 1 = lambda by 1 into p of zero when x = 1
P of 2 = lambda by 2 into p of 1 when x = 2 and so on.
If we have to find the theoretical frequencies, then multiply the probabilities by N, the total frequency.

Here's a summary of our learning in this session:

- Definition of binomial distribution
- Moments
- Nature of the distribution
- Moment and Cumulants Generating functions
- Nature of the distribution
- Additive property of the distribution
- Recurrence relation for probabilities
- Mode
- Fitting the distribution