Frequently Asked Questions

1. Define Poisson distribution?

Answer: A random variable X is said to follow Poisson distribution with parameter λ if its probability mass function is given by:

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad , x = 0, 1, 2, \ldots$$

= 0 otherwise and

we write X ~ P(λ)

2. Drive pmf of Poisson distribution.

Answer:

Poisson distribution is limiting case of binomial distribution under the following conditions:

(i) n, the number of trials is indefinitely large, that is n→∞(read as n tends to infinity)

(ii) P, the constant probability of success for each trial is indefinitely small, that is p→0.

(iii) np = λ, (say), is finite. Thus p = λ/n, q = 1 – λ/n, where λ is a positive real number.

The probability of x successes in a series of n independent trials is

$$p(x) = \binom{n}{x} p^{x} q^{n-x} \quad , x = 0, 1, 2, \ldots, n$$

We want the limiting form of p(x) under the above conditions. Hence

$$\lim_{n \to \infty} p(x) = \lim_{n \to \infty} \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^{x} \left[1 - \frac{\lambda}{n}\right]^{n-x}$$ (Substituting p in terms of λ)

Using Stirling’s approximation for n! as n tends to infinity, viz.,

$\lim_{n \to \infty} n! = \sqrt{2\Pi} e^{-n} n^{n+\frac{1}{2}}$, we get

$$\lim_{n \to \infty} p(x) = \lim_{n \to \infty} \left[\frac{\sqrt{2\Pi} e^{-n} n^{n+\frac{1}{2}}}{x! \sqrt{2\Pi} e^{-(n-x)} (n-x)^{(n-x)+\frac{1}{2}}} \right] \left(\frac{\lambda}{n}\right)^{x} \left[1 - \frac{\lambda}{n}\right]^{n-x}$$

$$= \frac{\lambda^x}{e^x x!} \lim_{n \to \infty} \left[\frac{n^{n-x+\frac{1}{2}}}{(n-x)^{(n-x)+\frac{1}{2}}} \right] \left[1 - \frac{\lambda}{n} \right]^{n-x}$$

(Cancelling e^{-n} and $\sqrt{2\pi}$ both in numerator and denominator and also taking λ^x , e^x and x! outside the limit and combining the powers of n)

$$= \frac{\lambda^x}{e^x x!} \lim_{n \to \infty} \left[\frac{\left(1 - \frac{\lambda}{n}\right)^{n-x}}{\left(1 - \frac{x}{n}\right)^{(n-x)+\frac{1}{2}}} \right]$$

(Taking $n^{n-x+(1/2)}$ in the denominator and cancelling this with the numerator in the above expression)

$$= \frac{\lambda^x}{e^x x!} \frac{\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{n} \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{-x}}{\lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^{n} \lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^{-x+\frac{1}{2}}}$$

, (splitting limits for each terms separately both in numerator and denominator)

We know that $\lim_{n \to \infty} \left(1 - \frac{a}{n}\right)^{n} = e^{-a}$.

Hence by applying this result in the above expression we get,

$$\lim_{n \to \infty} p(x) = \frac{\lambda^x}{e^x x!} \frac{e^{-\lambda} \times 1}{e^{-x} \times 1}$$

(as second terms in both numerator and denominator are independent of n in power)

$$= \frac{e^{-\lambda} \lambda^x}{x!} ; \; x = 0, 1, 2, \ldots, \infty$$

which is the probability function of the Poisson distribution and λ is known as the parameter of the distribution.

3. Give examples of Poisson distribution

Answer:

1. Number of deaths from a disease (not in the form of an epidemic) such as heart attack or cancer or due to snake bite.
2. Number of suicides reported in a particular city.

3. The number of defective material in a packing manufactured by a good concern.
4. Number of faulty blades in a packet of 100.
5. Number of air accidents in some unit of time
6. Number of printing mistakes at each page of the book.
7. Number of telephone calls received at a particular telephone exchange in some unit of time or connections to wrong numbers in a telephone exchange.
8. Number of cars passing a crossing per minute during the busy hours of a day.
9. The number of fragments received by a surface area 't' from a fragment atom bomb.
10. The emission of radioactive (alpha) particles

4. What is the relationship between mean and variance of Poisson distribution?

Answer: For Poisson distribution mean and variance are equal

5. Find mean of Poisson distribution

Answer:

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x p(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x+1-1}}{x(x-1)!}$$ (expanding x! as x(x-1)! and adding and subtracting 1 in the power of λ)

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right)$$

$$= \lambda e^{-\lambda}.e^{\lambda} = \lambda$$

Hence the mean of the Poisson distribution is λ

6. Obtain variance of Poisson distribution

Answer:

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x p(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x+1-1}}{x(x-1)!}$$ (expanding x! as x(x-1)! and adding and subtracting 1 in the power of λ)

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right)$$

$$= \lambda e^{-\lambda}.e^{\lambda} = \lambda$$

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} \left[x(x-1) + x\right] \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x-2+2}}{x(x-1)(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 .\ e^{\lambda} + \lambda = \lambda^2 + \lambda$$

Variance is given by,

$\mu_2 = \mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$

7. Explain the nature of the Poisson distribution.

Answer: Coefficient of skewness is given by,

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$

Hence the Poisson distribution is always a skewed distribution

Coefficient of Kurtosis is given by,

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda}$, which is greater than 3.

Hence Poisson distribution has leptokurtic curve.

8. Derive moment generating function of Poisson distribution.

Answer:
$M_X(t) = E\,(e^{tX})$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda}.e^{\lambda e^t}$$

$$= e^{-\lambda(1-e^t)}$$

9. State and prove reproductive property of Poisson distribution.

Answer

Statement: If X_i , (i = 1, 2, 3, . . n) are independent Poisson variates with parameters λ_i, (i = 1, 2, 3, . . n respectively then $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$

Proof

We know that $M_{X_i}(t) = e^{-\lambda_i(1-e^t)}$

Consider $M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$

$$= e^{-\lambda_1(1-e^t)}\, e^{-\lambda 2(1-e^t)} \ldots e^{-\lambda n(1-e^t)}$$

$$= e^{-(\lambda_1+\lambda_2\ldots\lambda_n)(1-e^t)}$$

Which is the mgf of Poisson distribution with parameter $\lambda_1 + \lambda_2 + \ldots \lambda_n$. Hence by uniqueness theorem of mgf, $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$

10. If X and Y are independent Poisson variates, show that the conditional distribution of X given X + Y is binomial.

Answer

Let X and Y be independent Poisson variates with parameters λ and μ respectively. Then X + Y is also a Poisson variate with parameter λ + μ.

$$P[X=r/(X+Y=n] = \frac{P(X=r \cap X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=r \cap Y=n-r)}{P(X+Y=n)}$$

$$= \frac{P(X=r)P(Y=n-r)}{P(X+Y=n)}$$, (since X and Y are independent)

$$= \frac{\frac{e^{-\lambda}\lambda^r}{r!} \cdot \frac{e^{-\mu}\mu^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!}}$$

$$= \frac{n!}{r!(n-r)!} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^r \left(\frac{\mu}{\lambda+\mu}\right)^{n-r}$$

$$= \binom{n}{r} p^r q^{n-r}$$, where p = λ/(λ+μ) and q = 1-p

Hence the conditional distribution of X given X+Y = n is a binomial distribution with parameters n and p = λ/(λ+μ)

11. Obtain mode of Poisson distribution

Answer
Mode is that value of x for which p(x) is maximum. Hence first we find the ratio of probabilities when X takes value x and x-1.

$$\frac{p(x)}{p(x-1)} = \frac{e^{-\lambda}\lambda^x}{x!} \bigg/ \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \frac{\lambda}{x}$$

Now let us consider the two cases such that λ is an integer and not an integer.

Case – 1: when λ is not an integer.

Let us suppose that λ can be written as m + f, where m is the integral part of λ and f is a fraction (<1). Then we have,

$\frac{p(1)}{p(0)} > 1$, $\frac{p(2)}{p(1)} > 1$, . . . , $\frac{p(m)}{p(m-1)} > 1$ -------------(1)

and $\frac{p(m+1)}{p(m)} < 1$, $\frac{p(m+2)}{p(m+1)} < 1$, . . . ---------------(2)

From (1) and (2) we can write,

p(0) < p(1) < p(2) , . . . <p(m-1) < p((m) >p(m+1) > p(m+2) > . . ., which shows that p(m) is the maximum value. Hence in this case the distribution is unimodal and m, the integral part of λ is the unique modal value.

Case – 2: when λ is an integer.

Let us suppose that λ can be written as m, an integer. Then we have,

$\frac{p(1)}{p(0)} > 1$, $\frac{p(2)}{p(1)} > 1$, . . . , $\frac{p(m-1)}{p(m-2)} > 1$ ------------- (1)

$\frac{p(m)}{p(m-1)} = 1$ --------------------------------------- (2)

and $\frac{p(m+1)}{p(m)} < 1$, $\frac{p(m+2)}{p(m+1)} < 1$, . . . --------------- (3)

From (1), (2) and (3) we can write,

$p(0) < p(1) < p(2), \ldots < p(m-1) = p((m) > p(m+1) > p(m+2) > \ldots$

In this case we have tow maximum values viz., p(m – 1) and p(m) and thus the distribution is bimodal and two modes are at m-1 and m. That is at λ – 1 and λ.

12. Derive Cumulants generating function of Poisson distribution

Answer:
Cumulants generating function (cgf) is obtained by taking logarithm of mgf and is denoted by $K_X(t)$.

$K_x(t) = \log M_X(t)$

$= \log e^{-\lambda(1-e^t)}$

$= -\lambda(1-e^t)$

$= -\lambda(1-(1+t+t^2/2! + t^3/3! + t^4/4!+ \ldots)$

$= \lambda(t + t^2/2! + t^3/3! + t^4/4!+ \ldots)$

13. Show that for Poisson distribution, all the Cumulants are equal

Answer:
Cumulants generating function is given by,

$K_x(t) = \lambda(t + t^2/2! + t^3/3! + t^4/4!+ \ldots)$

r^{th} Cumulants K_r is given by equating the coefficient of $t^r/r!$ in $K_x(t)$
K_1 = Coefficient of t in $K_X(t) = \lambda$ (the mean of the distribution)
K_2 = Coefficient of $t^2/2!$ in $K_X(t) = \lambda$ (the variance of the distribution μ_2)
K_3 = Coefficient of $t^3/3!$ in $K_X(t) = \lambda$
K_4 = Coefficient of $t^4/4!$ in $K_X(t) = \lambda$
Hence if we take any Cumulants, it is equal to λ.

14. Find the recurrence relation for the successive probabilities of Poisson distribution

Answer:

Let us consider,

$$\frac{p(x)}{p(x-1)} = \frac{e^{-\lambda}\lambda^{x}}{x!} \Big/ \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \frac{\lambda}{x}$$ Therefore the recurrence relation for the probabilities of Poisson distribution is given by,

p(x) = (λ/x).p(x-1)

15. Explain how will you fit a Poisson distribution to the given data?

Answer:

First we find the recurrence relation for the probabilities of Poisson distribution

Hence consider the ratio of probabilities when X takes value x and x-1

$$\frac{p(x)}{p(x-1)} = \frac{\lambda}{x}$$

Therefore p(x) = (λ/x).p(x-1)

First we find p(0) = $e^{-\lambda}$, where λ is estimated from the given data if it is not known using the estimator, sample mean. The other probabilities can be easily obtained by substituting x = 1, 2, 3, …

ie. p(1) = (λ/1).p(0), when x= 1

p(2) = (λ/2).p(1), when x= 2 and so on

If we have to find the theoretical frequencies, then multiply the probabilities by N, the total frequency.