1. Introduction

Welcome to the series of eLearning modules on Bernoulli Distribution.

By the end of this session, you will be able to:

- Explain Bernoulli distribution
- Explain various measures of Bernoulli distribution
- Explain the application of the Bernoulli distribution
- Explain the sum of independent Bernoulli variables

Now, let us discuss the Bernoulli Distribution.

Bernoulli distribution was discovered by a Swiss Scientist James Bernoulli (1654-1705) and was published in the year 1713 that is eight years after his death.

Definition:
A random variable X which assumes values 1 and 0 with respective probabilities p and q = 1-p is called Bernoulli variable.

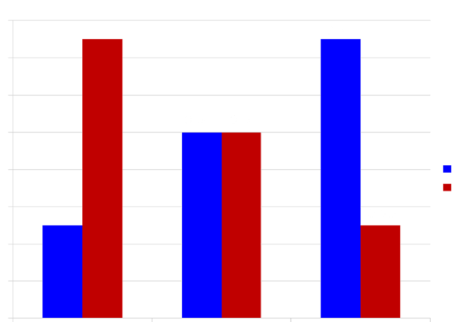
In other words, a random variable X is said to follow Bernoulli distribution with parameter 'p' if its probability mass function is given of the form,

(p of x equal to p to the power x into q to the power 1 minus x for x=0 and 1)
= 0 otherwise
Where, $0 < p < 1$ and $p + q = 1$
And we write as x follows Bernoulli distribution with parameter p

Following graph shows the Bernoulli distribution Probability Mass Function (PMF).

Distribution function

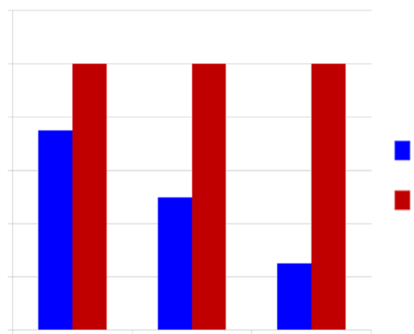
Figure 1



F(x) = (0 for x less than 0)
= (1-p for 0 less than or equal to x less than 1)
= (1 for x greater then equal to 1)
Following graph shows the cumulative distribution function for different values of p.

Figure 2

P	0.25	0.5	0.75
F(x)	0.75	0.5	0.25



2. Examples

Let us discuss the examples of Bernoulli Distribution.
Bernoulli random variables occur in a number of instances in practice. It is very useful for modelling an event that may or may not occur.

1. A fair coin is tossed. Let the random variable X take values 1 and 0 according as the toss results in “Head” or “Tail” respectively. Then X is a Bernoulli variate with parameter p=1/2. Here, X denotes the Head obtained in the toss.
2. Let the variable X take values 1 and 0 according as a newborn baby is “Male” or “Female” respectively. Then, X is a Bernoulli variate with parameter p=P (baby is male). Here, X denotes the newborn baby is male.
3. A bomb is dropped from an aeroplane on a bridge. Let the variable X take the values 1 and 0 according as the bomb “Succeeds” or “Fails” to hit the bridge. Then, X is Bernoulli Variate with p=P (bomb hits the bridge). Here, X is the number of bombs hitting the bridge.
4. Let the variable X take values 1 and 0 according as an industrial product tested is “Defective” or “Non-defective”. Then, X is Bernoulli variate with parameter p=P(the product is defective). Here, X is the defective product manufactured by the industry.

Likewise, we can give many examples for Bernoulli distribution, namely, Success of medical Treatment, Interviewed person is female, Student passes in an examination, Transmittance of a disease etc.

In fact, any random experiment can be dichotomised (which can be divided into only two classes) to yield a Bernoulli trial.

3. Mean and Variance of Bernoulli Distribution

Let us now, discuss Mean and variance of Bernoulli distribution.

If x follows Bernoulli distribution with parameter p, then probability distribution of X can be written as

Table 1

X	1	0
p(x)	p	q

(E of X equal to summation x into p of x)
= 1xp + 0xq
= p
Mean E(X) = p

Variance is given by (V of X equal to E of X square minus E of X whole square)

Let us consider,
(E of X square equal to summation x square into p of x)
= (1 square xp plus 0 square xq)
= p

Hence, $V(X) = p - p^2 = p(1\text{-}p) = pq$
Standard Deviation is given by (Root of v of x equal to root of pq)

Observe that for Bernoulli distribution is mean is greater than that of variance.

Let us now, discuss the Moment Generating Function (mgf).
(Mx of t equal to E of e to the power tx)
= (e to the power t into 1 xp plus e to the power t into 0 xq)
= (p into e to the power t plus q)

From moment generating function, we can find raw moments by differentiating (Mx of t equal to p into e to the power t plus q), with respect to x at t=0

(Single differentiation of mgf equal to p into e to the power t when t equal to 0 and μ_1 dash equal to p)
(Double differentiation of mgf equal to p into e to the power t when t equal to 0 and μ_2 dash equal to p)

(Triple differentiation of mgf equal to p into e to the power t when t equal to 0 and μ_3 dash equal to p)
(Fourth differentiation of mgf equal to p into e to the power t when t equal to 0 and μ_4 dash equal to p)

And hence we can obtain central moments
(μ_2 equal to pq)
(μ_3 equal to pq of q minus p)
(μ_4 equal to pq of 1 minus 3pq)

4. Skewness and Kurtosis of Bernoulli Distribution

Let us now discuss the Skewness
(beta 1 equal to µ3 square upon µ2 cube equal to q minus p whole square upon pq)
Observe that the skewness depends on the value of the parameter p.
Hence, Bernoulli distribution is negatively skewed if p > 0.5, positively skewed if p< 0.5 and if p=0.5, the distribution is symmetric.

Let us now discuss the Kurtosis
(beta 2 equal to µ4 upon µ2 square equal to 1 upon pq minus 3)

The kurtosis goes to infinity for high and low values of p. However, for p=1/2 the Bernoulli distribution has very low kurtosis than any other distribution.

Let us now discuss the Mean deviation from mean.

Mean deviation from mean is given by

(MD equal to E of absolute X minus E of X)

= (E of absolute X minus p)

= (Summation absolute x minus p into p of x)

= (Summation absolute x minus p into p to the power x into q to the power 1 minus x)

= (pq plus pq)

= (2pq)

<u>Result - 1</u>

If X has Bernoulli distribution with parameter p, then can show that 1-X is also a Bernoulli variate with parameter q.

We prove the above result by using Moment generating function of Bernoulli distribution.

We have (M 1-x of t equal to E of e to the power t of 1 minus x)
= (E of e to the power t into e to the power minus tx)
= (e to the power t into E of e to the power minus tx)
= (e to the power t into summation e to the power minus tx into p of x)
= (e to the power t into summation e to the power minus tx into p to the power

x into q to the power 1 minus x)

= (e to the power t of e to the power minus t into 0 into p to the power 0 into q to the power 1 minus 0 plus e to the power minus t into 1 into p to the power 1 into q to the power 1 minus 1)

= (e to the power t into q plus p)

This is the moment generating function of Bernoulli distribution with parameter q. Hence, by uniqueness theorem of mgf, 1-X has Bernoulli distribution with parameter q.

Result – 2

If X be a Bernoulli random variable with parameter 0.5, then we can show that tenth raw moment is also 0.5.

Given that, X is Bernoulli variate with parameter 0.5.

We know that the mgf of the distribution is (Mx of t equal to p into e to the power t)
To find 10^{th} raw moment we need to differentiate the mgf 10 times at t = 0
And all the derivative of (p into e to the power t plus q is p into e to the power t)
At t=0, tenth derivative become $pe^0 = p$

Hence, the tenth raw moment is p = 0.5

5. Problem Solving

Now let us, solve few problems.

Illustration-1

If X is a Bernoulli variate with $p = 0.45$, find the mean and variance. Also, find the probability that the variate takes the value zero.

Solution:

It is given that X is a Bernoulli variate and $p=0.45$, $q = 1 – p = 0.55$
Hence (p of x equal to p to the power x into q to the power 1 minus x at x equal to 0, 1)
= 0 otherwise

Mean, $p = 0.45$
variance, $pq = 0.45 \times 0.55 = 0.2475$

Probability that the variable takes value zero is given by substitution $x = 0$ in $p(x)$.

Therefore, (P of x equal to 0 equal to p to the power 0 into q to the power 1 minus 0 equal to 0.45 to the power 0 into 0.55 to the power 1 minus 0 equal to 0.55)

Illustration – 2

A biased coin is tossed, which has the probability of appearing head, 0.72. Find the probability that a tail occurs.

Solution:
Let a random variable X denotes head occurs in a single toss. Hence, X has Bernoulli distribution with probability of success p, 0.72 and $q = 1- p = 0.28$

Hence, (p of x equal to p to the power x into q to the power 1 minus x at x equal to 0, 1)

Probability of getting a tail means getting no head in a single toss.
Therefore, (P of x equal to 0 equal to p to the power 0 into q to the power 1 minus 0 equal to 0.72 to the power 0 into 0.28 to the power 1 minus 0 equal to 0.28)

Illustration – 3

In a city with four lakh voters, 125000 are in favour of a particular party. If a randomly selected voter favours particular party, then a random variable X assume value 1. Otherwise, let it take value zero. Find the mean and standard deviation of X.

Solution:
Given that out of four lakh voters, 125000 are in favour of a particular party and the variable takes value 1if a selected voter favours that party.

That is, $P(X=1)$ =(one lakh twenty five thousand by four lakh) = 0.3125

Hence, $X \sim$ Bernoulli (p), $p = 0.3125$ and $q = 1 - p = 0.6875$

Therefore mean of X, $p = 0.3125$
Variance of X, $pq = 0.3125 \times 0.6875 = 0.2148$ and standard deviation is given by, root of 0.2148 = 0.4635

Now, let us discuss the Sum of independent Bernoulli variables.

Let a Bernoulli experiment be conducted (repeated) n times. Let the variable X_i ($i=1, 2 \dots n$) take values 1 and 0 according to the i^{th} experiment is a success or a failure. Then X_i is Bernoulli variate with parameter p. It denotes the number of successes in the i^{th} experiment. Let $X = X_1 + X_2 + \dots + X_n$. Then X denotes the number of successes in these n repetitions.

Following cases give sum of independent Bernoulli Variables.

1. Let a coin be tossed 3 times. Let X_i ($i = 1,2,3$) be a variable which takes value 1 and 0 according as the i^{th} toss results in Head or Tail. Then, $X = X_1 + X_2 + X_3$ denote 'the number of heads' obtained in 3 tosses.
2. A dog gives birth to 5 puppies. Let X_i ($i = 1,2,3,4,5$) be a variable which takes values 1 and 0 according as the i^{th} puppy is male or female. Then $X = X_1 + X_2 + \dots + X_5$ denotes 'the number of male puppies' among the 5 puppies.
3. 8 bombs are dropped on a bridge. Let X_i ($i=1, 2, \dots, 8$) be a variable which takes values 1 and 0 according as the i^{th} bomb succeeds or fails to hit the bridge. Then, $X = X_1 + X_2 + \dots + X_8$ denotes 'the number of bombs hitting' the bridge among the 8 bombs.

Here's a summary of our learning in this session:

- Explain Bernoulli distribution
- Explain various measures of Bernoulli distribution
- Explain the application of the Bernoulli distribution
- Explain the sum of independent Bernoulli variables