1. Introduction

Welcome to the series of eLearning modules on Discrete Uniform Distribution.

By the end of this session, you will be able to:

- Explain discrete uniform distribution
- Explain various measures like mean, variance, median, mean deviation, moments, skewness and kurtosis of the discrete uniform distribution
- Explain the application of the discrete uniform distribution

In probability theory, the discrete uniform distribution is a probability distribution whereby, a finite number of equally spaced values are equally likely to be observed. Every one of *n* values has equal probability *1/n*. Another way of saying "discrete uniform distribution" would be "a known, finite number of equally spaced outcomes equally likely to happen."

The discrete uniform distribution is also known as the ‘equally likely outcomes’ distribution.

Let us take an example for discrete uniform distribution. Here, we would be rolling a fair 6-sided die.

Let X be the random variable denoting what number is thrown.

P(X = 1) = 1/6
P(X = 2) = 1/6 etc

In fact, P(X = x) = 1/6 for all x between 1 and 6. Hence, we have a discrete uniform distribution.

If two dice are thrown and their values added, the uniform distribution no longer fits since the values from 2 to 12 do not have equal probabilities.

2. Probability Mass Function

Probability Mass Function of Discrete Uniform Distribution

Let S be the set of integers, then probability mass function of the discrete uniform distribution having n values is given by,

$p(x)= 1/n,\ a < x < b$

$= 0$ otherwise

(a belongs to positive and negative integers, b belongs to positive and negative integers, x belongs to a, a+1, a+2 to b-1 and b)

In particular if we restrict the range to be the positive integers, taking values 1, 2, . . . , n then probability mass function of the discrete uniform distribution is given by,

$p(x) = 1/n,\ x = 1, 2, \ldots, n$

Let us now discuss the graph of probability mass function. If we plot the probability mass function of the discrete uniform distribution, we get the graph as follows.
Here n = 5

Now, let us discuss the cumulative distribution function.
For general case, cumulative distribution function is given by,

(F(x) equal to summation p of x equal to summation 1 upon n equal to absolute x minus a plus 1 upon n)

Hence, we can write

(absolute x minus a plus 1 upon n for a less than or equal to x less than or equal to b)
$= 1$ for $x > b$

If we plot the cumulative distribution function by taking n =5, we get,

Let us now, discuss the Mean of the distribution.

We can find mean (expectation) of the discrete uniform distribution as follows:

Let X has uniform distribution with probability mass function $P(x)= 1/n,\ a<x<b$ (only integer values between a and b).

(E of X equal to summation x into p of x)
=(a into p of x=a plus a plus 1 into p of x equal to a+1 to b minus 1 into p of x equal to b minus 1 plus b into p of x equal to b)
=(a into 1 upon n plus a plus 1 into 1 upon n to b minus 1 into 1 upon n plus b into 1 upon n)
=(a plus a plus 1 to b minus 1 plus b into 1 upon n)

In particular if we take, P(x)=1/n, x=1, 2, 3, . . ., n, then
(E of X equal to summation x into p of x)
= (1 into p of x equal to 1 plus 2 into p of x equal to 2 to n minus 1 into p of x equal to n minus 1 plus n into p of x equal to n)
=(1 into 1 upon n plus 2 into 2 upon n to n minus 1 into 1upon n plus n into 1 upon n)
=(1 plus 2 to n minus 1 plus n into 1 upon n)
=(n into n plus 1 upon 2 into 1 upon n)
=(n plus 1 upon 2)

3. Variance of the Distribution

Variance is given by (v of x equal to e of x square plus e of x whole square)
To find variance, first we consider the particular case and then generalise it because it is easy to work out with simple than that of general case.

If we take, P(x)=1/n, x=1, 2, 3, . . ., n, then

First let us find $E(X)^2$
(E of X square equal to summation x square into p of x)
=(1 square into p of x equal to 1 plus 2 square into p of x equal to 2 to n minus 1 whole square into p of x equal to n minus 1 plus n square into p of x equal n)
=(1 square into 1 upon n plus 2 square into 1 upon n to n minus 1 whole square into 1 upon n plus n square into 1 upon n)
=(1 square into 2 square to n minus 1 whole square plus n square into 1 upon n)

(n into n plus 1 into 2n plus 1 upon 6 into 1 upon n)
(n plus 1 into 2n plus 1 upon 6)

Hence (v of x equal to n plus 1 into 2n plus 1 upon 6 plus n plus 1 upon 2 whole square)
=(n square minus 1 upon 12)

If we apply in general formula, we know that n = b-a+1

Therefore in general, (v of x equal to b minus a plus 1 whole square minus 1 upon 12)

Median of the distribution

Median of the distribution is nothing but the middle most value. Hence in general median for discrete uniform distribution is given by
Median = a plus b by 2
if we take, P(x)=1/n, x=1, 2, 3, . . ., n, then
Median = n plus 1 by 2 (the median of discrete series with n values).

Mean Deviation about mean or median

if we take, P(x)=1/n, x=1, 2, 3, . . ., n, then mean deviation is given by,

(MD equal to 1 upon n summation absolute x minus half of n plus 1)

To do the sum, we consider separately the cases of n is odd and n is even.

If n is odd, that is we can write in general, n= 2m-1 then

(MD equal to 1 upon n summation absolute x minus m)

=1 upon n into summation m minus x plus summation x minus m)

=
Two m square minus 2m into n plus 1 plus n square plus n upon 2n)
=
n square minus 1 upon 4n)

If n is even, that is we can write in general, n= 2m then

MD equal to 1 upon n summation absolute x minus m plus half)
=
(1 upon n into summation m plus half minus x plus summation x minus of m plus half)
=
(n upon 2 plus m square upon n minus m)
=
(n upon 4)

Hence the complete solution is given by

(MD equal to n square minus 1 upon 4n for n odd)
= (n upon 4 for n even)

4. Moment Generating Function

We can now discuss the Moment Generating Function

(Mx of t equal to E of e to the power tx)

=

(Summation e to the power tx into p of x)

=

(1 upon n into e to the power t minus e to the power t of n plus 1 upon 1 minus e to the power t)

=

(e to the power t of 1 minus e to the power tn upon n of 1 minus e to the power t)

Let us now discuss the Moments of the distribution

If we take, P(x)=1/n, x=1, 2, 3, . . ., n, then
Raw moments (moments about zero) are given by,
(µ r dash equal to E of X to the power r equal to summation x to the power y into p of x equal to 1 upon n summation x to the power y)

Therefore

(µ 1 dash equal to E of X equal to 1 upon n summation x equal to n into n plus 1 upon 2n equal to n plus 1 upon 2)
(sum of n natural numbers is n into n plus one by two)
(µ 2 dash equal to E of X square equal to 1 upon n summation x square equal to n into n plus 1 into 2n plus 1 upon 6n equal to n plus 1 into 2n plus 1 upon 6)
(sum of squares of n natural numbers is n into n plus one into two n plus one by six)
(µ 3 dash equal to E of X cube equal to 1 upon n summation x cube equal to n square into n plus 1 whole square upon 4n equal to n into n plus 1 whole square upon 4)
(sum of cubes of n natural numbers is n into n plus one by two whole square)

(µ 4 dash equal to E of X to the power 4 equal to 1 upon n summation x to the power 4 equal to n into n plus 1 into 2n plus 1 into 3n square minus 3n minus 1 upon 30n equal n plus 1 into 2n plus 1 into 3n square minus 3n minus 1 upon 30)
(sum of 4^{th} power of n natural numbers is n into n plus one into two n plus one into three n squarer minus three n minus one by thirty)

Now, the Central moments (moments about mean) are given by

(µ r equal to E of X to the power r minus E of X whole to the power r equal to summation x minus E of X whole to the power y into P of x equal to 1 upon n summation x minus E of X whole to the power y)

and the moments about the mean are

(µ 2 equal to n square minus 1 upon 12)

(μ 3 equal to zero)

(μ 4 equal to 1 upon 240 into n square minus 1 into 3n square minus 7)

Having discussed the central moments, we can now discuss the skewness and kurtosis of the distribution.

Skewness is given by

(Beta one equal to μ 3 square upon μ 2 cube equal to 0)

Hence the discrete uniform distribution is symmetric.

Kurtosis is given by
(beta 2 equal to μ 4 upon μ 2 square equal to 3 into 3n square minus 7 upon 5 into n square minus 1)

which is less than 3
Hence, the discrete uniform distribution has Platikurtic curve.

5. Application of Discrete Uniform Distribution

Let us now, discuss the application of discrete uniform distribution.

It rare, that we come across a variable that can take one of the several values each with equal probability. It is used in solving ‘German tank problem” occurred during world war II.

In wartime, a key goal of military intelligence is to determine the strength of an enemy force. In World War II, the Western Allies wanted to estimate the number of tanks the Germans had.

The use of serial number analysis was not only to estimate production but also used to understand German production more generally. This included number of factories, relative importance of factories, length of supply chain (based on lag between production and use), changes in production, and use of resources such as rubber.

To estimate the number of tanks produced up to a certain point, the Allies used the serial numbers on tanks. Most significantly, gearbox numbers fell in two unbroken sequences; also chassis and engine. Though these were more complicated, various other components helped crosscheck the analysis. Similar analyses were done on tires that were observed to be sequentially numbered (i.e. 1, 2, 3... *N*).

The minimum-variance unbiased estimator is given by

(n equal to m into 1 plus 1 upon k into minus 1)

Where, **m** is the largest serial number observed (sample maximum) and *k* is the number of tanks observed (sample size).

Suppose there are 15 tanks, numbered 1, 2, 3... 15. An intelligence officer has spotted tanks 2, 6, 7, 14. Using the above formula, the values for *m* and *k* would be 14 and 4, respectively. The formula gives a value 16.5, which is close to the actual number of tanks, 15. Then suppose the officer spots an additional 2 tanks, neither of them #15. Now *k* = 6 and *m* remains 14. The formula gives a better estimate of 15.333...

Here’s a summary of our learning in this session:

- Explain discrete uniform distribution
- Explain various measures like mean, variance, median, mean deviation, moments, skewness and kurtosis of the discrete uniform distribution
- Explain the application of the discrete uniform distribution