1. Introduction & Usefulness of Moment Generating Function

Welcome to the series of E-learning modules on moment generating functions or mgf, its uses, properties and limitations. Here we have also proved the uniqueness of mgf and using mgf we have obtained Cumulant generating function and hence Cumulants.

At the end of this session, you will be able to:

- Understand about moment generating function
- Understand the uses of moment generating functions
- Describe the properties of moment generating functions
- Describe the limitations of moment generating functions
- Explain Uniqueness theorem and
- Obtain moments from Cumulant generating function

Introduction

In probability theory and statistics, the moment generating function of a random variable is an alternative specification of its probability distribution. Thus, it provides the basis of an alternative route to analytical results compared with working directly with probability density functions or cumulative distribution functions.

There are particularly simple results for the moment-generating functions of distributions defined by the weighted sums of random variables.

- Note, however, that not all random variables have moment-generating functions
- In addition to univariate distributions, moment-generating functions can be defined for vector- or matrix-valued random variables, and can even be extended to more general cases

Now let us discuss the usefulness of moment generating function.

Moment generating function is useful in many ways:

(a) They provide an easy way of calculating the moments of a distribution

(b) They provide some powerful tools for addressing certain counting and combinatorial problems

(c) They provide an easy way of characterizing the distribution of the sum of independent random variables

(d) They provide tools for dealing with the distribution of the sum of a random number of independent random variables

(e) They play a central role in the study of branching processes

(f) They play a key role in large deviations theory, that is, in studying the asymptotics of tail probabilities of the form $P(X \geq c)$, when c is a large number

(g) They provide a bridge between complex analysis and probability, so that complex analysis methods can be brought to bear on probability problems

(h) They provide powerful tools for proving limit theorems, such as laws of large numbers and the central limit theorem

2. Moment Generating Functions

The moment generating function or m g f. of a random variable X (about origin) having the probability density function f of (x) is given by,
M X of t is equal to expectation of e power t X
Is equal to integral of e power t x into f of x d x

The integration being extended to the entire range of x, t being the real parameter and it is being assumed that the right hand side of the moment generating function is absolutely convergent for some positive number h such that minus h less than t less than h

Thus M X of t is equal to expectation of e power t X
Is equal to expectation of one plus t into X plus t square into X square divided by two factorial plus etc., plus t power r X power r divided by r factorial plus etc.
Is equal to one plus t into expectation of X plus t square by two factorial into expectation of X square plus and so on plus, t power r divided by r factorial into Expectation of X power r plus etc.
Is equal to q plus t into mu one dash plus t square by two factorial into mu two dash plus etc., plus t power r by r factorial into mu r dash plus etc.
Is equal to summation over r is equal to zero to infinity t power r divided by r factorial into mu r dash

Where mu r dash is equal to integral over x, x power r into f of x d x is the r^{th} moment of X about origin. Thus the coefficient of t power r divided r factorial in M X of (T) gives mu r dash (about origin). Since M X of (t) generates moments, it is known as moment generating function.

Given the moment generating function we can also find moments by differentiating it with respect of t, r times and then putting t is equal to zero.
That is, r th derivative d power r d t power r M X of t at t is equal to zero
Is equal to mu r dash divided by r factorial into r factorial plus mu r plus one dash into t plus mu r plus two dash into t square divided by two factorial plus etc., at t is equal to zero
Implies, mu r dash is equal to d power r by d t power r M X of t at t is equal to zero.

In general, the moment generating function X about the point X is equal to 'a' is defined as,
M X of t about X is equal to 'a'
Is equal to Expectation of 'e' power t into X minus A
Is equal to Expectation of one plus t into X minus a plus t square into X minus 'a' the whole square divided by two factorial plus etc. plus t power r into X minus 'a' power r divided by r factorial plus and so on.
Is equal to one plus t into mu one dash plus t square divided by two factorial into mu two dash plus etc., plus t power r divided by r factorial into mu r dash plus and so on.
Where mu r dash is equal to Expectation of X minus a whole power r is the r_{th} moment about the point X is equal to 'a'.

3. Limitations of MGF

Now let us discuss some of the limitations of moment generating function.
Moment generating function suffers from some drawbacks which have restricted its use in Statistics. Below are the some of the deficiencies of moment generating function .We have discussed these in detail in the first semester.

i. A random variable X may have no moments although its mgf exists
ii. A random variable X can have moment generating function and some (or all) moments, yet the moment generating function does not generate the moments
iii. A random variable X can have all or some moments but mgf does not exist except perhaps at one point

Consider the following illustration
Show that for the following distribution with probability density function f of x is equal to theta into a power theta divided by x power theta plus one where x is greater than or equal to a and theta is greater than one, the moments of all order exists but moment generating function does not exist.

Proof
Now consider r^{th} raw moment
Mu r dash is equal to expectation of x power r is equal to theta into 'a' power theta into integral over 'a' to infinity x power r minus theta minus one d x. This is equal to theta into 'a' power theta into x power r minus theta divided by r minus theta, ranges from 'a' to infinity
This is equal to theta into a power r divided by theta minus r, where theta is greater than r.
Hence r^{th} raw moment exists.

Now let us consider the moment generating function of the given distribution.
M X of t is equal to expectation of 'e' power t X is equal to theta into 'a' power theta into 'e' power t x divided by x power theta plus one d x, Which does not exist. Since 'e' power t into x dominates x power theta plus one and (e power t into x divided by x power theta plus one) tends to infinity and hence the integral is not convergent. That is moment generating function for the given distribution does not exist.

4. Properties of MGF

Properties

Now let us discuss some of the properties of moment generating function which holds good for continuous as well as discrete random variables.

The first property is stated as follows.

M c into X of (t) is equal to M X of (c into t), c being a constant.

Let us prove this as follows.

By Definition, left hand side

Is equal to M c into X of (t) is equal to Expectation of (e power t into (c into X))

Is equal to Expectation of (e power (t into c) into X)

Is equal to M X of c into t which is equal to right hand side.

Hence the proof.

Second property is given as follows.

The moment generating function of the sum of the independent random variables is equal to the product of their respective moment generating functions. Symbolically, if x one, x two, x three, etc till x n are independent random variables, then the moment generating function of their sum x one plus x two plus x three and so on till plus x n is given by,

M x one plus x two plus etc., plus x n of (t) is equal to M X one of (t) into M X two of (t) into etc., into M X n of (t)

Proof: By definition,

M x one plus x two plus and so on till plus x n of (t) is equal to

Expectation of e power t into x one plus X two plus and so on till plus X n)

Is equal to Expectation of (e power t into X one plus t into X two plus etc., plus t into X n)

Is equal to Expectation of e power t into X one into Expectation of e power t into X two into etc into Expectation of (e power t into X n)

Is equal to M X one of t into M X two of t into etc., into M X n of t.

Third property is about the effect of change of origin and scale on moment generating function.

Let us transform X to the new variable U by changing both the origin and scale in X as follows.

U is equal to X minus a divided by h, where 'a' and 'h' are constants.

Moment generating function of U (about origin) is given by

M U of t is equal to expectation of 'e' to the power t into U

Is equal to Expectation of {exponential of [t into (x minus a) divided by h]}

Is equal to Expectation of (e to the power 't' into x divided by h into e to the power minus 'a' into 't' divided by 'h')

Is equal to 'e' power minus 'a' into 't' divided by 'h' into Expectation of e power t into X divided by h

Is equal to e power minus 'a' into t divided by h into Expectation of e power X into t by h

Is equal to e power minus 'a' into t divided by h into M u of t by h.

Where M X of (t) is the moment generating function of X about origin.

In particular, if we take ‘a’ is equal to Expectation of (X) is equal to mu and h is equal to sigma X is equal to sigma then U is equal to [X minus Expectation of (X)] divided by sigma X is equal to (X minus mu) divided by sigma is equal to Z is known as a standard variate. Thus moment generating function of a standard variate Z is given by,
M Z of t is equal to e power minus mu into t divided by sigma into M X of t divided by sigma

Form the above note that,
Expectation of Z is equal to expectation of X minus mu divided by sigma is equal to one by sigma into expectation of X minus mu
This is equal to one by sigma into expectation of x minus mu
This is equal to one divided by sigma into mu minus mu which is equal to zero.
And variance of Z is equal to Variance of X minus mu divided by sigma
Is equal to one by sigma square into Variance of X minus mu
Is equal to one divided by sigma square into Variance of X
Is equal to one divided by sigma square into sigma square, which is equal to one.
Therefore expectation of Z is equal to zero and Variance of Z is equal to one. That is, mean and variance of a standard variate are 0 and one respectively.

5. Uniqueness Theorem

Now let us consider the uniqueness theorem of moment generating function.
The moment generating function of a distribution, if it exists, uniquelly determines the distribution.
This implies that corresponding to a given probability distribution, there is only one moment generating function (provided it exists) and corresponding to given moment generating function there is only one probability distribution.
Hence M X of (t) is equal to M Y of (t) implies X and Y are identically distributed.

Proof
If X and Y are two random variables with probability density functions f of (x) and f of (y) respectively. Let moment generating function of two distributions are given by, M X of (t) and M Y of (t) respectively. We need to prove that if M X of t is equal to M Y of (t) then f of (x) is equal to f of (y).

We know that moment generating function of X and Y are given by
M of (t) is equal to Expectation of e power t into X is equal to integral over X e power t into X f of x d x and
M Y of (t) is equal to Expectation of (e power t into Y) is equal to integral over Y e power t into Y f of (y) d y
If M X of (t) is equal to M Y of (t) implies,
integral over X e power t into X f of x d x
is equal to integral over Y e power t into Y f of (y) d y
Which implies f of (X) is equal to f of (y)
That is X and Y are identically distributed.

Cumulant generating function of the distribution is obtained from moment generating function and is denoted by K X of (t) and is given by,
K X of (t) is equal to log M X of (t) to the base e, provided the right hand side of the equation can be expanded as a convergent series in powers of t.

Thus
K X of t is equal to k one into y plus k two into t square divided by two factorial plus and so on till, plus k r into t power r divided by r factorial plus and so on , is equal to log M X of t to the base 'e'.
This is equal to log mu one dash into t plus mu two dash into t square divided by two factorial plus etc till, plus mu r dash into t power r divided by r factorial plus etc.
Where k r is equal to coefficient of t to the power r divided by r factorial in K X of (t) is called rth Cumulant.

Hence
K one into t plus k two into t square divided by two factorial plus k three into t cube by three factorial plus k four into t power four divided by four factorial plus and so on,
Is equal to mu one dash into t plus mu two dash into t square divided by two factorial plus mu three dash into t cube divided by three factorial plus mu four dash into t power four divided by four factorial

Minus half into mu one dash into t plus mu two dash into t square divided by two factorial plus mu three dash into t cube divided by three factorial plus mu four dash into t power four divided by four factorial
Plus one divided by three into mu one dash into t plus mu two dash into t square divided by two factorial plus mu three dash into t cube divided by three factorial plus mu four dash into t power four divided by four factorial
Minus one divided by four into mu one dash into t plus mu two dash into t square divided by two factorial plus mu three dash into t cube divided by three factorial plus mu four dash into t power four divided by four factorial
Plus and so on

Comparing the powers of t on both sides, we get the relationship between the moments and Cumulants. Hence we have
K one is equal to mu one dash is equal to Mean
K two divided two factorial is equal to mu two dash divided by two factorial plus mu one dash square divided by two factorial
This implies, k two is equal to mu two dash plus mu one dash square is equal to mu two.
K three divided by three factorial is equal to mu three dash divided by three factorial minus half into mu one dash into mu two dash divided by two factorial plus mu one dash cube divided by three
Which implies k three is equal to mu three dash minus three into mu two dash into mu one dash plus two into mu one dash cube is equal to mu three.

Also
K four divided by four factorial is equal to mu four dash divided by four minus half into mu two dash square divided by four plus two into mu one dash into mu three dash divided by three factorial plus one divided by three into three into mu one dash square into mu two dash divided by two factorial plus mu one dash to the power four divided by four.
On simplification we get,
k four is equal to mu four dash minus four into mu three dash into mu one dash plus six into mu two dash into mu one dash square minus three into mu one dash to the power four minus three into mu two dash minus mu one dash square the whole square.
Hence k four is equal to mu four minus three k two square
Implies mu four is equal to k four into three k two square.

Here's a summary of our learning in this session where we have:

- Understood about moment generating functions
- Understood the uses of moment generating functions
- Explained the Properties and limitations of mgf
- Understood the Uniqueness theorem
- Understood how to obtaining moments from Cumulant generating function