Frequently Asked Questions

1. Write a note on Moment Generating Function (mgf).
   Answer:
   In probability theory and statistics, the moment generating function of a random variable is an alternative specification of its probability distribution. Thus, it provides the basis of an alternative route to analytical results compared with working directly with probability density functions or cumulative distribution functions. There are particularly simple results for the moment-generating functions of distributions defined by the weighted sums of random variables.

2. Define mgf.
   Answer:
   The moment generating function (mgf) of a random variable X (about origin) having the probability density function f(x) is given by, $M_X(t) = E(e^{tX}) = \int e^{tx} f(x) dx$

3. Obtain expression of mgf in terms of raw moment.
   Answer:

$$M_X(t) = E(e^{tX}) = E\left(1 + tX + \frac{t^2 X^2}{2!} + \ldots + \frac{t^r X^r}{r!} + \ldots\right)$$

$$= 1 + tE(X) + \frac{t^2}{2!} E(X^2) + \ldots + \frac{t^r}{r!} E(X^r) + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

4. Mention the uses of mgf.
   Answer:
   Moment generating function is useful in many ways:

   They provide an easy way of calculating the moments of a distribution.

   - They provide some powerful tools for addressing certain counting and combinatorial problems
   - They provide an easy way of characterizing the distribution of the sum of independent random variables
   - They provide tools for dealing with the distribution of the sum of a random number of independent random variables
   - They play a central role in the study of branching processes
   - They play a key role in large deviations theory, that is, in studying the asymptotics of tail probabilities of the form P(X ≥ c), when c is a large number

- They provide a bridge between complex analysis and probability, so that complex analysis methods can be brought to bear on probability problems

They provide powerful tools for proving limit theorems, such as laws of large numbers and the central limit theorem.

5. How to find moments from moment generating function using calculus?
Answer:
Given the mgf we can also find moments by differentiating it with respect to t r times and then putting t=0.
That is, $\left|\frac{d^r}{dt^r} M_X(t)\right|_{t=0} = \left|\frac{\mu_r'}{r!}.r!+\mu_{r+1}'t + \mu_{r+2}'\frac{t^2}{2!} + \ldots\right|_{t=0} \Rightarrow \mu_r' = \left|\frac{d^r}{dt^r} M_X(t)\right|_{t=0}$

6. From the above definition obtain mgf about the point X=a.
Answer:
$$M_X(t)(X = a) = E(e^{t(X-a)})$$
$$= E\left(1 + t(X-a) + \frac{t^2(X-a)^2}{2!} + \ldots + \frac{t^r(X-a)^r}{r!} + \ldots\right)$$
$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots$$
Where $\mu_r'=E[(X-a)^r]$, is the r^{th} moment about the point X=a

7. What are the limitations of mgf.
Answer:
Moment generating function suffers from some drawbacks which have restricted its use in Statistics. Below are the some of the deficiencies of mgf, which we discussed in detail in the first semester.
   i. A random variable X may have no moments although its mgf exists
   ii. A random variable X can have mgf and some (or all) moments, yet the mgf does not generate the moments
   iii. A random variable X can have all or some moments but mgf does not exist except perhaps at one point

8. Show that mgf does not exist even if all order moment exists.
Answer:
This we prove using an example.
Consider the distribution with pdf $f(x) = \frac{\theta a^\theta}{x^{\theta+1}}; x \ge a, \theta > 1$

Hence r^{th} raw moment is given by,

$$\mu_r' = E(X^r) = \theta a^\theta \int_a^\infty x^{r-\theta-1}dx = \theta a^\theta \left|\frac{X^{r-\theta}}{r-\theta}\right|_a^\infty = \frac{\theta a^r}{\theta - r}; \theta > r$$

Hence r^{th} raw moment exists.

Now let us consider the mgf of the given distribution.

$M_X(t) = E(e^{tX}) = \theta a^\theta \int_a^\infty \frac{e^{tx}}{x^{\theta+1}} dx$, which does not exist, since e^{tx} dominates $x^{\theta+1}$ and $(e^{tx}/x^{\theta+1}) \rightarrow \infty$ and hence the integral is not convergent. That is mgf for the given distribution does not exist.

9. Write the properties of mgf.
Answer:
The properties of mgf are,
   a. $M_{cX}(t)=M_X(ct)$, c being a constant
   b. The moment generating function of the sum of the independent random variables is equal to the product of their respective moment generating functions. Symbolically, if X_1, X_2, X_3,…,X_n are independent random variables, then the moment generating function of their sum $X_1+X_2+X_3+…+X_n$ is given by, $M_{X1+X2+…+Xn}(t)=M_{X1}(t).\ M_{X2}(t)…\ M_{Xn}(t)$
   c. Effect of change of origin and scale on mgf – if U=(X-a)/h then $M_U(t)= e^{-at/h} M_U(t/h)$

10. Show that $M_{cX}(t)=M_X(ct)$, c being a constant.
Answer:
By Definition,

L.H.S= $M_{cX}(t)=E(e^{t(cX)})$

$= E(e^{(tc)X})$

$= M_X(ct)$ = R.H.S

Hence the proof.

11. Show that the moment generating function of the sum of the independent random variables is equal to the product of their respective moment generating functions.
Answer:
X_1, X_2, X_3,…,X_n are independent random variables, then the moment generating function of their sum $X_1+X_2+X_3+…+X_n$ is given by,

$M_{X1+X2+…+Xn}(t)= E(e^{t(X1+X2+…+Xn)})$

$= E(e^{tX1+tX2+…tXn})$

$= E(e^{tX1})\ E(e^{tX2})…\ E(e^{tXn})$

$=M_{X1}(t).\ M_{X2}(t)…\ M_{Xn}(t)$

12. What is the effect of change of origin and scale on mgf?
Answer:
Let us transform X to the new variable U by changing both the origin and scale in X as follows.
U=(X-a)/h, where a and h are constants.
Mgf of U (about origin) is given by

$M_U(t)=E(e^{tU})=E\{exp[t(x-a)/h]\}$

$= E(e^{tX/h}e^{-at/h})$

$=e^{-at/h} E(e^{tX/h})$

$= e^{-at/h} E(e^{X(t/h)}) = e^{-at/h} M_U(t/h)$

Where $M_X(t)$ is the mgf of X about origin.

13. Obtain the mgf of standard variate.

Answer:
We know that, $M_U(t)= e^{-at/h} M_U(t/h)$
In particular, if we take $a=E(X) =\mu$ and $h=\sigma_X=\sigma$ then $U=[X-E(X)]/\sigma_X =(X-\mu)/\sigma=Z$ is known as a standard variate. Thus mgf of a standard variate Z is given by,
$M_Z(t)=e^{-\mu t/\sigma} M_X(t/\sigma)$

14. State uniqueness theorem of mgf.
Answer:
The moment generating function of a distribution, if it exists, uniquely determines the distribution.

This implies that corresponding to a given probability distribution, there is only one mgf (provided it exists) and corresponding to given mgf there is only one probability distribution. Hence $M_X(t)=M_Y(t)$ implies X and Y are identically distributed.

15. Give the expression for Cumulant generating function.
Answer:
Cumulant generating function of the distribution is obtained from moment generating function and is denoted by $K_X(t)$ and is given by,
$K_X(t)=\log_e M_X(t)$