1. Introduction

Welcome to the series of E-learning modules on Expectation of a Random Variable and its properties, addition theorem, multiplication, expectation of a constant, expectation of product of function of random variables, expectation of linear combination of random variables and variance and its important properties.

By the end of this session, you will be able to :

- Understand the meaning of expectation of continuous random variables
- Understand the condition for existence of expectation of a random variable
- Understand Addition theorem of expectation
- Describe Multiplication theorem of expectation
- Explain expectation of linear combination of random variables.
- Explain expectation of functions of random variables.
- Understand Variance and its properties

Many frequently used random variables can be both characterized and dealt effectively for practical purposes by consideration of quantities called their expectation. For example, a gambler might be interested in his average winning at a game.

A businessman in his average profits on a product, a physicist in the average charge of a particle and so on.

The 'average' value of a random phenomenon is also termed as its mathematical expectation or expected value.

Expected value :
Once we have constructed the probability distribution for a random variable, we often want to compute the mean or expected value of the random variable.

The expected value of a continuous random variable, X with probability density function f(x), is as follows
Expectation of X is equal to integration from minus infinity to infinity x into f of x d x.
Provided the right hand integral or series is absolutely convergent.
That is,
Integral over minus infinity to infinity modulus of x into f of x is equal to integral over minus infinity to infinity, modulus of x into f of x d x is finite.

2. Notes on Random Variables, Properties of Expectation of a Random Variable

Consider the following notes on random variables.

- Since absolute convergence implies ordinary convergence, when expectation of mod X has finite value, then, expectation of X is also finite.
- Expectation of X exists, if Expectation of mod X exists

Expected value and variance of an indicator variable:
Consider the indicator variable X is equal to I A so that
X is equal to I A is equal to one if A happens and 0 if A dash or A complement happens.
Now Expectation of (X) is equal to one into Probability of (X is equal to one) plus zero into Probability of (X is equal to zero)
Implies, Expectation of (I_A) is equal to one into Probability of (I_A is equal to one) plus zero into Probability of (I_A is equal to zero)
Therefore Expectation of (I_A) is equal to Probability of (A)

This gives us a very useful tool to find P(A), rather than to evaluate E(X)
Thus Probability of (A) is equal to Expectation of (I_A)
Expectation of (X square) is equal to one square into Probability of (X is equal to 1) plus zero square into Probability of (X is equal to zero) is equal to Probability of (I_A is equal to one) is equal to Probability of (A).
Hence Variance of (X) is equal to Expectation of (X square) minus [Expectation of (X)] the whole square is equal to Probability of (A) plus probability of A square
Is equal to Probability of (A) into [one minus Probability of (A)]
Is equal to Probability of (A) into Probability of (A complement.)

Consider the next note.
Let us consider the random variable X with probability density function
f of x is equal to one divided by phi into one by one plus x square, where x ranges from minus infinity to infinity.
To check whether expectation exists, let us find expectation of modulus of X
Which is equal to integral from minus infinity to infinity, mod x into f of x d x
Is equal to one divided by phi into integral from minus infinity to infinity, mod x divided by one plus x square d x
Since above is an even function, we can write integral from minus infinity to infinity as two times zero to infinity.
Hence we get, two divided by phi into integral over zero to infinity x divided by one plus x square d x.

Integrating by substitution, we get
One divided by phi into log one plus x square, ranges from zero to infinity, Which tends to infinity.

Since this integral does not converge to a finite limit, Expectation (X) does not exist.

Now let us study the different properties of expectation of a random variable one by one as follows.

The first property gives the addition theorem of expectation.
If X and Y are random variables, the expectation of X plus Y is equal to Expectation of X plus expectation of Y.
Let two continuous random variables be X and Y and Let the joint probability density function of X Y is given by f of (x, y). Let f of (x) and f of (y) denotes the marginal probability density function of X and Y respectively.

Let expectation of X be equal to integral over x into f of x d x and expectation of Y be equal to integral over y into f of y d y.
Where f of x and f of y are the marginal density functions of X and Y respectively.
Now consider
Expectation of X plus Y is equal double integral over x and y, x plus y into f of x, y dy dx is equal to integral over x, x into integral over y, f of x, y dy dx plus integral over y, y into integral over x, f of x, y dx dy

This is equal to integral over x, x into f of x d x plus integral over y, y into f of y d y
Is equal to expectation of X plus Expectation of Y

We can generalize this property for n random variables as follows
Expectation of (x one plus X_2 plus etc., plus x n) is equal to Expectation of (x one) plus Expectation of (x two) plus etc., plus Expectation of (x n)
That is, expectation of summation x 'i' is equal to summation expectation of X i, if all expectations exists.
This we prove by induction.
From above, we can write,
Expectation of (x one plus x two) is equal to Expectation of (x one) plus Expectation of (x two)

Let us suppose that the above property is true for r random variables so that
Expectation of summation over i is equal to one to r X I is equal to summation I is equal one to r, expectation of X i.
Let expectation of summation over i is equal to one to r plus one X i is equal to expectation of summation over I is equal to one to r X I plus X r plus one
Is equal to expectation of summation over I is equal to one to r X i plus Expectation of X r plus one
Is equal to summation over I is equal to r expectation of X I plus Expectation of X r plus one
Is equal to expectation of I is equal to one to r plus one, expectation of X i.
Hence result holds good for r+ one random variables. Hence by the principle of mathematical induction the result is true for all positive integral values of n.

3. Properties 2 and 3

Second property of random variable gives the multiplication theorem of expectation.
For n independent variables, expectation of product of variables is same as the product of the expectation of random variables.
That is Expectation of (X1 into X2 into etc., X_n) is equal to Expectation of (X_1) into expectation of (X_2) into etc., into expectation of (X_n)
Proof:
We prove this property also for two independent random variables, X and Y.

Let the joint probability density function of X, Y is given by f of (x, y). Let f of xx) and f(y) denotes the marginal probability density functions of X and Y respectively.
Let Expectation of X be equal to integral over x into f of x d x and expectation of Y be equal to integral over y into f of y d y.
Where f of x and f of y are the marginal density functions of X and Y respectively.

Now consider
Expectation of X into Y is equal to integral over x, integral over x, x into y f of x, y dx dx
Since X and Y are independent, we get joint probability function is the product of individual probability functions. That is f of x, y is equal to f of x into f of y.
Hence we get integral over x, integral over y, x into y f of x into f of y d x d y.

As we can split the x and y functions separately, we write,
Expectation of X into Y is equal to integral over x, x into f of x d x into integral over y, y into f of y d y, which is same as expectation of X into expectation of Y.
Hence in general, for n independent random variables,
Expectation of (X one into X two into etc till x n) is equal to Expectation of (x one) into expectation of (x two) into etc., into expectation of (x n)
That is expectation of product of n independent random variables is the product of expectation of individual random variables.

Third property is stated as follows.
If X is a random variable and 'a' is constant then
Expectation of [a into psi of (X)] is equal to a into Expectation of [psi of (X)]
Expectation of [psi of (X) plus a] is equal to Expectation of [psi of (X)] plus a
Where psi of (X), a function of X, is a random variable and all the expectations exist.

This property is proved as follows.
Consider the left hand side of (i)
That is, expectation of a into psi of X
Is equal to integral over x a into psi of x into f of x dx
Is equal to a into psi of x into f of x dx
Is equal to 'a' into expectation of psi of X.

Now consider the left hand side of (ii)
Expectation of psi of x plus a
Is equal to integral over x psi of x plus a into f of x d x

Is equal to integral over x psi of x into f of x d x plus a into integral over x f of x d x
Is equal to Expectation of [psi of (X)] plus a, since integral over x, f of x d x is equal to one.

Now consider some of the notes on property three.
If psi of (X) is equal to X in property three, we get
Expectation of (a into X) is equal to a into Expectation of (X) and
Expectation of (X plus a) is equal to Expectation of (X) plus a
If psi of (X) is equal to one, then Expectation of (a) is equal to a.

4. Properties 4, 5 and 6

If X is a continuous random variable with probability density function f of (x) and ‘a’ and ‘b’ are constants, then expectation of (a into X plus b) is equal to a into Expectation of (X) plus b

We prove above property as follows.
Expectation of ‘a’ into x plus ‘b’ is equal to integral over x, ‘a’ into x plus b f of x d x
Is equal to integral over x, ‘a’ into x into f of x d x plus integral over ‘b’ into f of x d x
Is equal to ‘a’ into integral over x, x into f of x dx plus b into integral over x f of x d x.
Observe that the first term is E(X) and the second integral is one. Therefore,
Expectation of (‘a’ into X plus b) is equal to ‘a’ into Expectation of (X) plus b.

Now consider some notes on property 4.
From Property 4,

- If b is equal to zero, Expectation of (a into X plus zero) is equal to a into Expectation of (X) plus zero is equal to a into Expectation of (X)
- Taking ‘a’ is equal to one and ‘b’ is equal to x bar, we get, Expectation of (X minus X bar) is equal to zero.
- The above property is defined only for the linear function. If the function is non linear, this result does not hold good.

Fifth property states the result of expectation on linear combination.
Let x one, x two, etc till x n be any n random variables and if a one, a two, etc till ‘a’ n are n constants, then
Expectation of summation over I from one to n a I into x I is equal to summation over I, from one to n, a I into expectation of X i.
Provided all the expectation exists.
The proof of this property follows from property one and 4.

Property 6 of expectation is given as follows.
If X is greater than or equal to zero then expectation of X is also greater than or equal to zero.

We prove above property as follows.
If X is a continuous random variables such X is greater than or equal to zero, then
Expectation of X is equal to integral from minus infinity to infinity x into f of x d x is equal to integral from zero to infinity x into f of x d x, which is greater than zero (since we have been given X is greater than or equal to zero and hence f of x is equal to zero if X is less than zero), provided expectation exists.

5. Properties 7, 8 and 9

Seventh property states that,
If X and Y are two random variables such that Y is less than or equal to X, then Expectation of (Y) is less than or equal to Expectation of (X)
We prove this property as follows.
Since Y is less than or equal to X, we have the random variable,
Y minus X is less than or equal to zero or X minus Y is greater than or equal to zero
Hence Expectation of (X minus Y) is greater than or equal to zero, implies, Expectation of (X) minus Expectation of (Y) is greater than or equal to zero.
Implies Expectation of (X) is greater than or equal to Expectation of (Y) or Expectation of (Y) is less than or equal to Expectation of (X)
Hence the property.

Eight property is stated as follows
Modulus of Expectation of (X) is less than or equal to Expectation of mod X, provided the expectation exists.
Proof:
Since X is less than or equal to mod X, we have from property six,
Expectation of (X) is less than or equal to Expectation of mod X – consider it as (1)
Again minus X is less than or equal to mod X, we have from property six,
Expectation of (minus X) is less than or equal to Expectation of mod X or minus Expectation of (X) is less than or equal to Expectation of mod X – consider it as (2)
From (1) and (2), we get,
Mod Expectation of (X) is less than or equal to Expectation of mod X

Property nine is stated as follows.
If X and Y are independent variables, then
Expectation of [h of (X) into k of (Y)] is equal to Expectation of [h of (X)] into Expectation of [k of (Y)], where h is a function of X alone and k is a function of Y alone, provided expectations on both sides exist.

Let us prove the property as follows
Let f of (x) and g of (y) be the marginal probability density functions of X and Y respectively.
Since X and Y are independent their joint probability density function f of (x, y) is given by
f of (x, y) is equal to f of (x) into f of (y)

By definition of continuous random variables,
Expectation of h of X into k of Y is equal to
Integral over y, integral over x h of x into k of y into f of x, y d x d y
Is equal to integral over y integral over x h of x into k of y into f of x into9 g of y d x d y

Since Expectation of [h of (X) into k of (Y)] exists, the integral on the right-hand side is absolutely convergent and hence by Fuibini's theorem for integrable functions, we change the order of integration to get,
Expectation of h of x into g of y is equal to integral over x h of x into f of x d x into integral over y k of y into g of y d y.

Is equal to expectation of h of X into Expectation of g of Y
Hence we prove the property.

Now let us discuss about the variance and its properties.
If X is a random variable, then variance of a random variable is given by,
Variance of (X) is equal to Expectation of (X square) minus [Expectation of (X)] the whole square.
The variance of a distribution satisfies the following properties.

i. Variance of (a into X plus b) is equal to a square into Variance of (X)
ii. Variance of (a into X plus or minus b into Y) us equal to a square into Variance of (X) plus b square into Variance of (X) plus or minus 2 into a into b into Covariance of (X, Y)

Here's a summary of our learning in this session where we have :

- Understood the meaning of expectation of continuous random variables
- Understood the condition for existence of expectation of a random variable
- Explained the Addition theorem of expectation
- Understood Multiplication theorem of expectation
- Explained about Expectation of linear combination of random variables.
- Understood Expectation of functions of random variables
- Understood Variance and its properties