Frequently Asked Questions

1. What do you mean by mathematical expectation?
   Answer:
   Many frequently used random variables can be both characterized and dealt effectively for practical purposes by consideration of quantities called their expectation. For example, a gambler might be interested in his average winning at a game, a businessman in his average profits on a product, a physicist in the average charge of a particle and so on. The 'average' value of a random phenomenon is also termed as its mathematical expectation or expected value.

2. Define mathematical expectation for continuous random variables.
   Answer:
   The expected value of a continuous random variable, X with probability density function f(x), is $E(X) = \int_{-\infty}^{\infty} xf(x)dx$, provided the right hand integral or series is absolutely convergent.

3. What do you mean by absolute convergence?
   Answer:
   Absolute convergent means $\int_{-\infty}^{\infty} |xf(x)|dx = \int_{-\infty}^{\infty} |x|f(x)dx$ should be finite.

4. What is expected value and variance of an indicator variable?
   Answer:
   Consider the indicator variable $X=I_A$ so that
   $X=I_A= 1$ if A happens
   0 if A' happens.
   Now E(X) = 1.P(X=1) + 0.P (X=0)
   Implies, E (I_A) = 1.P $(I_A=1)$ + 0.P $(I_A=0)$
   Therefore E (I_A)=P(A)
   This gives us a very useful tool to find P(A), rather than to evaluate E(X)
   Thus P (A) = E(I_A)
   $E(X^2) = 1^2 .P (X=1) + 0^2 .P (X=0) = P (I_A=1) = P(A)$
   $Var (X) = E(X^2) - [E (X)]^2 = P(A) + [P (A)]^2$
   = P (A) [1-P (A)] = P(A) P (A')

5. Check whether expectation exists for the random variable X with the following probability density function $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}; -\infty < x < \infty$
   Answer:
   To check whether expectation exists, let us find
   $E|X| = \int_{-\infty}^{\infty} |x|f(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|}{1+x^2}dx = \frac{2}{\pi}\int_{0}^{\infty} \frac{x}{1+x^2}dx = \frac{1}{\pi}\left|\log(1+x^2)\right|_0^{\infty} \to \infty$
   Since this integral does not converge to a finite limit E(X) does not exist.

6. State and prove addition theorem for two continuous random variables.
   Answer:
   If X and Y are random variables, then E(X+Y)=E(X)+E(Y)

Proof:

Let two continuous random variables be X and Y and let the joint pdf of XY is given by F (x y)

Let f(x) and f(y) denotes the marginal p.d.f of X and Y respectively.

Let E(X) = $\int_x xf(x)dx$ and E(Y) = $\int_y yf(y)dy$, where f(x) and f(y) are the marginal density function of X and Y respectively.

Consider $E(X+Y) = \int_x \int_y (x+y)f(x,y)dydx$

$$= \int_x x\left[\int_y f(x,y)dy\right]dx + \int_y y\left[\int_x f(x,y)dx\right]dy$$

$$= \int_x xf(x)dx + \int_y yf(y)dy = = E(X)+E(Y)$$

7. State and prove multiplication theorem.

Answer:

For n independent variables, expectation of product of variables is same as the product of the expectation of random variables.

i.e., $E(X_1X_2 \ldots X_n) = E(X_1)\, E(X_2) \ldots E(X_n)$

Proof:

This property also we prove for two independent random variables, X and Y. Let the joint p.d.f of XY is given by f(x, y). Let f(x) and f(y) denotes the marginal p.d.f of X and Y respectively.

Let E(X)= $\int_x xf(x)dx$ and E(Y) = $\int_y yf(y)dy$, where f(x) and f(y) are the marginal density function of X and Y respectively.

$$E(XY) = \int_y \int_x (xy)f(x,y)dxdy$$

$= \int_x \int_y (xy)f(x)f(y)dxdy$ [Since X and Y are independent,

f(x, y)=f(X) f(y)]

As we can split the x and y functions separately, we write,

$$E(XY) = \left[\int_x xf(x)dx\right]\left[\int_y yf(y)dy\right] = E(X)E(Y)$$

Hence in general, for n independent random variables,

$E(X_1X_2 \ldots X_n) = E(X_1)\, E(X_2) \ldots E(X_n)$

8. If X is a random variable and 'a' is constant then show that

i. E[aΨ(X)] =a E [Ψ(X)]

ii. E [Ψ(X)+a]=E[Ψ(X)]+a, where Ψ(X), a function of X, is a random variable and all the expectations exist.

Answer:
Consider the left hand side of (i)

$$E[a\Psi(X)] = \int_x a\Psi(x)f(x)dx = a\int_x \Psi(x)f(x)dx = aE[\Psi(X)]$$

Now consider the left hand side of (ii)

$$E[\Psi(X)+a] = \int_x [\Psi(x)+a]f(x)dx = \int_x \Psi(x)f(x)dx + a\int_x f(x)dx$$

$$= E[\Psi(X)]+a \quad (\because \int_x f(x)dx = 1)$$

9. If X is a continuous random variable with p.d.f f(x) and 'a' and 'b' are constants, then show that E(a X + b) =a E (X) + b

   Answer:

$$E(aX+b) = \int_x (ax+b)f(x)dx = \int_x (ax)f(x)dx + \int_x (b)f(x)dx$$

$$= a\int_x (x)f(x)dx + b\int_x f(x)dx$$

   Observe that the first term is E(X) and the second integral is 1. Therefore,
   E (a X + b) =a E (X) + b

10. Let $X_1, X_2, \ldots X_n$ be any n random variables and if $a_1, a_2, \ldots a_n$ are n constants, then show that $E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$, provided all the expectation exists.

Answer:
The proof of this result follows from results of 6 and 9.

11. If X≥0 then show that E(X)≥0

Answer:
If X is a continuous variable, such that X≥0, then $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{\infty} xf(x)dx > 0$
[Since we have given X ≥0 and f(x)=0 if X<0] provided expectation exists.

12. If X and Y are two random variables such that Y≤X, then E(Y)≤E(X)

Answer:
Since Y≤X, we have the random variable,
Y-X≤0 or X-Y≥0
Hence E(X-Y)≥0, implies, E(X)-E(Y)≥0
Implies E(X)≥ E(Y) or E(Y)≤E(X)
Hence the proof.

13. If X and Y are two random variables such that Y≤X, then E(Y)≤E(X)

    Answer:
    Since Y≤X, we have the random variable,
    Y-X≤0 or X-Y ≥ 0
    Hence E(X-Y) ≥ 0, implies, E (X) - E(Y) ≥ 0
    Implies E(X) ≥ E(Y) or E(Y) ≤ E(X)
    Hence the proof.

14. Show that |E(X)|≤E|X|, provided the expectation exists.

Answer:

Since X ≤ |X|, we have from property 6,

E(X) ≤ E|X|----------- (1)

Again -X ≤ |X|, we have from property 6,

E (-X) ≤ E|X| or -E(X) ≤ E|X|--------- (2)

From (1) and (2), we get,

|E(X)| ≤ E|X|

15. If X and Y are independent variables, then show that E [h (X).k(Y)]=E[h(X)].E[k(Y)], where h(.) is a function of X alone and k(.) is a function of Y alone, provided expectations on both sides exist.

Answer:

Let f(x) and g(y) be the marginal p.d.f's of X and Y respectively. Since X and Y are independent their joint p.d.f f(x, y) is given by

f(x, y) = f(x) . f(y)

By definition of continuous random variables,

$$E[h(X).k(Y)] = \int_y \int_x h(x)k(y)f(x,y)dxdy \; = \int_y \int_x h(x)k(y)f(x)g(y)dxdy$$

Since E[h(X)k(Y)] exists, the integral on the right-hand side is absolutely convergent and hence by Fuibini's theorem for integral functions, we change the order of integration to get,