1.Introduction

Welcome to the series of e-learning module on Axiomatic Probability. In this module we will cover the concept of axiomatic definition of probability.

By the end of this module, you will be able to explain the following:
- Probability as a set function
- Axiomatic definition of probability

The modern theory of probability is based on the axiomatic approach introduced by A.N.KOLMOGOROV a Russian mathematician.

This approach focuses on introducing probability as a set function. The approach also lays down some concepts, properties or postulates known as axioms based on which the entire theory is developed.

Axiomatic Probability is defined as – given a sample space of a random experiment, the probability of the occurrence of any event A is defined as a set function P(A) satisfying the axioms.

Let us try to understand certain concepts before studying the axiomatic definition in detail Sample space & Event in this context.

## 2.Sample space

Sample space is the set of all exhaustive cases of the random experiment. The outcomes are known as Sample Points.

Mathematically if e1, e2, e3,……, en are mutually exclusive outcomes then set S= (e1, e2, e3,……, en) sample space.
Each event is the outcome of the experiment.

Any repetition of experiment results in outcome corresponding to one & only one of the events in S.

Number of sample point in S is denoted as n(s).

If a coin is tossed at random, the sample space is S= (H, T) and n(S) =2.

If two coins are tossed at random then sample space S = {(H, T) X (H,T)} = {(HH, HT, TH,TT} hence n(S)= 4 which is $2^2$.

If three coins are tossed then sample space S= {(H, T) X (H, T) X (H, T)} = {(HHH, HTH, THH, TTH, HHT, HTT, THT, TTT}hence n(S) = 8 which is $2^3$.
In general, in a random toss of N coins, n(S)= 2N

# 3. Event

Event – of all the possible outcomes in the sample space of a random experiment, some out comes satisfy a specified description, which we call an event.

In a toss of three coins the sample space S={(H,T) X (H,T) X (H,T)} = {(HHH, HTH, THH,TTH, HHT, HTT, THT, TTT} = { w1, w2, w3, w4, w5,w6,w7, w8}

n(S)= 8=$2^3$
E1 = event of getting all heads = (HHH) = (w1)
E2 = event of getting exactly two heads (HTH, THH, HHT) = (w2, w3, w5)
E3 = event of getting at least two heads = (w1, w2, w3, w5) = (w1) Ù (w2, w3, w5) =E1 Ù E2.

Every event is expressed as a disjoint union of the single element subsets of S, hence the algebra of sets is used.

Events as sets : If A and B are two events then,
AÙB = an event which represents the happening of at least one of the events A and B.

For example if A={1,2,3,4}, B={3,4,5,6} then AÙB ={1,2,3,4,5,6}
AÇB = an event which represents the simultaneous happening of both the events A and B.

In the above example • AÇB={3,4}
Ā = A does not happen.
Ā ÇB = neither A nor B happens.
Ā Ç B = A does not happen but B happens.
(A ÇB) Ù (Ā Ç B) = exactly one of the two events A and B happens

The relationship between events and corresponding sample space can be represented graphically by using a Venn diagram. A Venn diagram is a graphical representation of complex probabilistic events using the rectangles and circles. The sample space is represented using the rectangle and the circles within the rectangle are representation of the events. The outside rectangle contains an area = 1 (representing all the possible cases).

Explanation for fig 1 – is a representation of the sample space S having three events A,B and C this figure also helps us in understanding the various operations between the events
A�B ( Event A intersection B) = regions 1 & 2, B�C (event B intersection C)=regions 1 & 3,AÙC (event A union C)= region 1,2,3,4,5,&7, B�A = (event B dash intersection A) = region 4&7, A�B�C = (event A intersection B intersection C) = region 1
(AÙB)�C = (event A union B intersection C dash = region 2, 6 &7.

For a better understanding let us take the fig 2 in this we can see that A, B & C are all sub sets of the sample space S. it also evident that event B is a subset of event A therefore event B intersection C has no element and hence are mutually exclusive events whereas event A intersection C has one element and A union B = to A.

## 4. Axiomatic approach to probability theory

Axiomatic approach to probability theory – Let S be a sample space, P be a real valued function defined on the subsets of S. Then P is called the probability function and the probability of event A is given by P(A) if the following three axioms hold good.

Axiom 1
0 ▯ P (A) ▯ 1
States that the probability that the outcome of the experiment is an outcome in A is some number between 0 and 1.

Axiom 2
P (S) = 1
States that with the probability 1, the outcome will be a point in the sample space S.

Axiom 3
If A1, A2,… is a sequence of mutually exclusive event then P(A1ÙA2Ù ….) = P(A1) +P(A2) + …….

States that for any sequence of mutually exclusive events the probability of at least one of these events occurring is the sum of their respective probabilities.

Let us look at some illustration to understand the Axioms in a better way.

Illustration 1
A coin is tossed thrice. What is the probability that at least one head occurs.
Solution:
Sample space S = (HHH, HTT, THT, TTH, HHT, HTH, THH, TTT)

If the coin is unbiased, each of these outcomes would be equally likely outcomes. A probability or weight w can be assigned to each sample point.

Then, 8w = 1 ▯w=1/8.

If A represents the event of at least one head occurring then,
A =(HHH, HTT, THT,TTH, HHT, HTH, THH) ▯P(A) = 1/8+1/8+1/8+1/8+1/8 +1/8+1/8 = 7/8.

On the other hand if the coin were to be bias and we feel that a head were twice as likely to appear as a tail, then we would have:

$P(\{H\}) = 2/3 \quad P(\{T\}) = 1/3$.

Illustration 2

If our experiment consists of tossing a coin and if we assume that a head is likely to appear as a tail, then we would have:

$P(\{H\}) = P(\{T\}) = \frac{1}{2}$

Illustration 3

Two coins are tossed together. Find the probability of getting head on both the coins.

Solution:

S = {HH, HT, TH, TT}

A = { HEAD ON BOTH COINS}

 n(A) = number of favourable sample points = 1

n(S) = number of possible sample points = 4

$P(A) = n(A) / n(S) = \frac{1}{4}$

Illustration 4

If a die is rolled and we suppose that all six sides are equally likely to appear

Solution:

S = {1,2,3,4,5,6}

A = {appearing of any number one to six}

n(A) = number of favourable points = 1

n(S) = number of sample points = 6

$P(A) = 1/6 = P\{1\}= P\{2\}= P\{3\}= P\{4\} =P\{5\}= P\{6\}$

A coin is successively tossed three times. Find the probability of getting:

1. Exactly one head
2. Exactly two heads

Solution:

S ={HHH, HTT, THT,TTH, HHT, HTH, THH} = n(S) = 8

1. If A is the event of getting exactly one head then,

a. A = {HTT, THT, TTH} = n(A) = 3, the required probability = $P(A) = 3/8$

3. Exactly one head or two heads.

2. If B is the event of getting exactly two heads, then,

a. B = { HHT, HTH, THH} = n (B) = 3, the required probability = $P(B) = 3/8$.

3. If C is the event of getting exactly one or two heads, then

a. C={HTT, THT, TTH, HHT, HTH, THH } = n(C ) = 6, the required probability = $P(C) = 6/8 = 3/4$ .

## 5.Propositions or Theorems using Axioms

Here on let us prove some simple propositions or theorems using the three Axioms.

Proposition 1: P(A) = 1 – P(A)
States that the probability that an event does not occur is 1 minus the probability that it does occur.
Since A and A are complimentary events, they are disjoint events.
Moreover, A A = S
From axioms 2 and 3, it follows that,
P (A A) = P(A) +P(A) = P(S) =1
P(A) = 1-P(A)

Proposition 2: If B A then, ( if event B is a subset of event A then)
i. P(AÇB) = P(A) – P(B) (probability of event A intersection B dash = Probability of event A minus probability of event B
ii. P(B) £ P(A) (also probability of event B is less than or equal to probability of event A)

Here we draw a Venn diagram which represents S as the sample space and A is an event and B is a subset of A and B is the event of B not occurring.
i. As B A, (B is the subset of A)
B and A Ç B are mutually exclusive events and their union is given by A.

Therefore, we have, P(A) =P[B (A Ç B)]

Using Axiom 3,
P(A) = P(B) + P(A Ç B)
Thus, P(A Ç B) = P(A) – P(B)
ii. Using Axiom 1,
P(A Ç B) 0
P(A) – P(B) 0
Hence,
P(B) P(A)

Proposition 3: For any two events A and B
P (AÇB) = P(B) – P(AÇB)
States that the probability of exactly one of the two events A and B happens is the probability of B minus probability of A and B.
A Ç B and A Ç B are disjoint events and

(AÇB)   (AÇB) = B

Thus, using axiom 3, we get,

P(B) = P (AÇB) + P (AÇB)

P (AÇB) = P(B) – P (AÇB)

Similarly, it can be proved that,

P (AÇB) = P(A) –P (AÇB)

Proposition 4: Probability of an impossible event is zero P (f) = 0

As we know, an impossible event contains no sample point thus the certain event S and impossible event are mutually exclusive.

It follows that

S         = S

Or P(S      ) = P(S)

From Axiom 3

P(S) + P( ) = P(S)

P( ) = 0

Proposition 5: For any two events A and B, proof that

P(AÇB)   P(AÇB) = P(A) + P (B) – 2 P(AÇB)

By the Venn diagram,

B = (AÇB) U (AÇB)

=>    P(B) = P(AÇB) + P(AÇB)

Now, P(AÇB) = P(B) - P(AÇB)     …….(*)

(from theorem 1)

P (BÇA) = P(A) – P(AÇB)    ……..(**)

(from theorem 1)

Adding (*) and (**),

P(AÇB) + P(AÇB) = P(A) + P(B) – 2P (AÇB)

or

P(AÇB)    P(AÇB) = P(A) + P(B) – 2P(AÇB)