1. Introduction

Welcome to the series of E-learning modules on Negative Binomial distribution. In this module we are going cover the definition of Negative Binomial distribution, moments, nature of the distribution, additive property, mgf, cgf and the limiting distribution.

By the end of this session, you will be able to explain:

- About Negative Binomial distribution
- Moment generating function
- Cumulants generating function
- Moments and Cumulants
- Nature of the distribution
- Limiting distribution of Negative Binomial distribution
- Additive Property of the distribution

The equality of the mean and variance is an important characteristic of the Poisson distribution, whereas for the binomial distribution the mean is always greater than the variance.

Occasionally, however, observable phenomena give rise to empirical discrete distributions which show a variance larger than the mean, which is a distinguishing feature of the Negative Binomial distribution.

The Negative Binomial distribution is sometimes also referred to as the Pascal distribution after the French mathematician Blaise Pascal (1623-1662), but there seems to be no historical justification.

Suppose we have a succession of *n* Bernoulli trails. We assume that:

i. The trials are independent
ii. The probability of success p in a trial remains constant from trial to trail

Let p of x denote the probability that there are x failures preceding the rth success in x plus r trials.

Now, the last trial must be a success, whose probability is *p*. In the remaining x plus r minus 1 trials we must have r minus 1 success whose probability is given by,
X plus r minus 1 c r minus 1 p power r minus 1 into q power x.

Therefore, by compound probability theorem, p of x is given by the product of these two probabilities
That is X plus r minus 1 c r minus 1 p power r minus 1 into q power x into p is equal to x plus r minus 1 c r minus 1 into p power r into q power x.

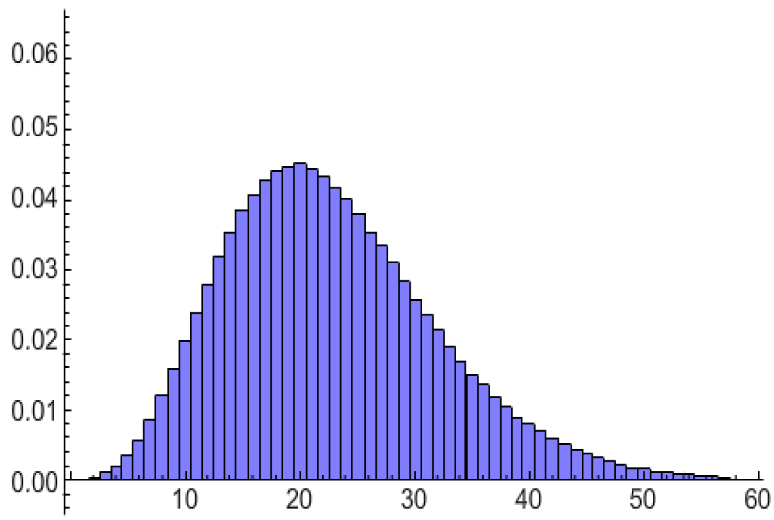
<u>Definition</u>
A random variable x is said to follow Negative Binomial distribution with parameters r and p, if its probability mass function is given by,

P of x is equal to x plus r minus 1 c r-1 into p power r into q power x , x take values 0, 1, 2, etc., and p lies between zero and one
And equal to zero otherwise.

We write x follows N B of r comma p, read as x follows Negative Binomial distribution with parameters r and p.

The graph of probability mass function for n is 8 and p is 0.26 is given as follows:

Figure 1



Negative Binomial distribution can be written as the binomial distribution with negative index as follows:

We know that x plus r minus 1 c r minus 1 is same as x plus r minus 1 c x.
Expanding using factorial and simplifying, we get,
X plus r minus 1 into x plus r minus 2 into etc into r plus 1 into r divided by x factorial.
Observe that there are x terms and taking minus 1 outside from each term, we get,
Minus 1 whole power x into minus r into minus r minus 1 into etc., into minus r minus x plus 2 into minus r minus x plus 1 whole divided by x factorial.
Which is equal to minus 1 power x into minus r c x.

Hence p of x is equal to minus r c x into p power r into minus q power x, where x take values zero, 1, 2 etc.
And zero otherwise.

This is the x plus 1th term in the expansion of p power r into 1 minus q whole power minus r, a binomial expansion with a negative index. Hence the distribution is known as Negative Binomial distribution.

Also summation over x from zero to infinity p of x is equal to p power r into summation over x from zero to infinity minus r c x minus q power x
which is equal to p power r into 1 minus q power minus r which is equal to 1 since 1 minus q

is p.

Therefore, p of x represents the probability function and the discrete variable which follows this probability function is called the Negative Binomial Variate.

If we put p is equal to 1 by Q and q is equal to P by Q so that Q minus P is equal to 1 then, P of x is equal to minus r c x, into Q power minus r into minus P by Q whole power x, x take values, zero, 1, 2, etc. And zero otherwise.
This is the general term in the Negative Binomial expansion Q minus P whole power minus r.

Suppose we take, r is equal to 1 in the probability mass function of the Negative Binomial distribution, then we get,
P of x is equal to q power x into p, x take values zero, 1, 2 etc., which is the probability function of geometric distribution.

Hence, Negative Binomial distribution can be regarded as the generalisation of the geometric distribution.

2. Moment Generating Function

Now let us find the moment generating function of the Negative Binomial distribution.
M X of t is equal to expectation of e power t x
Is equal to summation over x from zero to infinity, e power t x into p of x
Is equal to summation over x e power t x into minus r c x into Q power minus r into minus P by Q whole power x.
Is equal to summation over x, minus r c x into Q power minus r into minus P into e power t by Q whole power x
Is equal to Q minus P into e power t whole power minus r.

We can obtain Mean and Variance of the distribution by differentiating the mgf with respect to t and then equating at t is equal to zero.

Hence, mew 1 dash is equal to d by dt of m x of t at t is equal to zero.
Is equal to minus r into minus P e power t, into Q minus P e power t power minus r minus 1 at t is equal to zero.
On simplification we get, r into P
Therefore, mean of the distribution is r P.

To find variance we first find mew 2 dash.
Mew 2 dash is equal to d square by d t square m x of t
Is equal to r into P e power t into Q minus P e power t whole power minus r minus 1, plus minus r minus 1 into r into P into e power t into Q minus P e Power t whole power minus r minus 2 into minus P into e power t at t is equal to zero
Is equal to r into P plus r into r plus 1 into p square
Therefore mew 2 is equal to mew 2 dash minus mew 1 dash whole square
Is equal to r into r plus 1 into P square plus r into P minus r square into P square, Which is equal to r into P into Q.

As Q is greater than 1, r into P is less than r into P into Q
That is mean is less than variance.

3. Cumulant Generating Function

Having moment generating function, let us find Cumulant generating function.

Cumulants generating function is obtained by taking logarithm of mgf and is denoted by k x of t.

Therefore, K x of t is equal to log M x of t
Is equal to log Q minus P into e power t whole power minus r
Is equal to minus r into log Q minus P into e power t
Using the exponential expansion for e power t we get,
Minus r into log Q minus P into 1 plus t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 by 4 factorial plus etc.
Is equal to minus r into log 1 minus P into t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 by 4 factorial plus etc.

By comparing this with Cumulants generating function of binomial distribution and proceeding likewise, we can find first four Cumulants as well as 2nd 3rd and 4th central moments.

First Cumulant k 1 , same as mean is equal to r P
K 2, same as variance is equal to r P Q
K 3, same as mew 3 is equal to r P into 1 plus 3 P plus 2 P square.
On simplification, we get, r P Q into Q plus P
And fourth Cumulant k 4 is equal to r P into 1 plus P into 1 plus 6P plus 6 p square.
On simplification we get,
R P Q into 1 plus 6 P Q
Therefore mew 4 is equal to k 4 plus 3 k 2 square
Substituting for k4 and k2 and then simplifying we get
r P Q into 1 plus 3 P Q into r plus 2.

Since Q is equal to1 by p and P is equal to q by p, we can write mean and central moments in terms of p and q.
Mean is equal to r into q by p
Variance mew 2 is equal to r into q by p square
Mew 3 is equal to r q into 1 plus q whole divided by p cube
And mew 4 is equal to r q into p square plus 3 q into r plus 2 whole divided by p power 4.

From above central moments, we can now find the nature of the distribution.

Skewness of the distribution is given by,
Beta 1 is equal to mew 3 square by mew 2 cube
Is equal to 1 plus q whole square divided by r into q
Hence, the Negative Binomial distribution is positively skewed.

Kurtosis is given by,
Beta 2 is equal to mew 4 divided by mew 2 square

Is equal to p square plus 3 q into r plus 2 divided by r q
Observe that beta 2 is greater than 3 and hence, Negative Binomial distribution has leptokurtic curve.

4. Recurrence Relation for the Central Moments

Now let us find the recurrence relation for the central moments of Negative Binomial distribution.

Let X is a Negative Binomial variate with probability mass function.
P of x is equal to x plus r minus 1 c x into p power r into q power x, x take values zero, 1, 2, etc.
And zero otherwise.

Also we know that the mean of the distribution is r into q by p.
Therefore mew k is equal to expectation of x minus r into q by p whole power k.
That is mew k is equal to summation over x from zero to infinity x minus r into q by p whole power k into p of x.
Is equal to summation over x, x minus r into q by p whole power k into x plus r minus 1 c x into p power r into q power x.

Differentiating with respect of q, we get
D mew r by d q is equal to summation over x k into x minus r q by p whole power k minus 1 into d by d q of x minus r q by p into r plus x minus 1 c x into q power x into p power r,
Plus summation over x, x minus r q by p whole power k into r plus x minus 1 c x into x into q power x minus 1 into p power r plus q power x into r into p power r minus 1 into d p by d q.

But d p by d q is equal to d by d q of 1 minus q is equal to minus 1 and
d by d q of x minus r q by p
is equal to d by d q of x minus r into 1 by p minus 1.
Is equal to r by p square into d p by d q is equal to minus r by p square.

Therefore,
D mew k by d q is equal to
Minus k r by p square into summation over x, x minus r q by p whole power k minus 1 into p of x, plus summation over x, x minus r q by p whole power k into r plus x minus 1 c x into q power x minus 1 into p power r into x minus r q by p.
Is equal to k r by p square mew k minus 1 plus 1 by q into summation over x, x minus r q by p whole power k plus 1 into p of x
Is equal to k r by p square into mew k minus 1 plus 1 by q into mew k plus 1

Implies, mew k plus 1 is equal to q into d mew k by d q plus k r by p square into mew k minus 1, k take values 1, 2, 3 etc.

5. Limiting Distribution of Negative Binomial Distribution

Now let us find the limiting distribution of Negative Binomial distribution.
Negative Binomial distribution tends to Poisson distribution as P tends to zero and r tends to infinity such that r P is equal to lambda, which is finite. Proceeding to the limits, we get
Limit p of x is equal to limit x plus r minus 1 c r minus 1 into p power r into q power x
Is equal to x plus r minus 1 c x, Q power minus r into P by Q whole power x.
Is equal to limit r tends to infinity, x plus r minus 1 into x plus r minus 2 into etc., r plus 1 into r divided by x factorial into 1 plus P whole power minus r into P divided by 1plus P whole power x.

Is equal to limit r tends to infinity 1 by x factorial into 1 plus x-1divided by r into 1 plus x minus 2 divided by r into etc., 1 plus 1 by r into 1 into r power x into 1 plus P whole power minus r into P divided by 1plus P whole power x.

Is equal to 1 by x factorial limit r tends to infinity 1 plus P whole power minus r into r P divided by 1plus P whole power x

Since, rP is equal to lambda, on substitution and simplification we get,
Lambda power x by x factorial into limit r tends to infinity, 1 plus lambda by r whole power minus r into limit r tends to infinity 1 plus lambda by r whole power minus x.

By applying the limits we get,
Lambda power x by x factorial into e power minus lambda into 1
Is equal to e power minus lambda, lambda power x by x factorial
which is the probability mass function of Poisson distribution with parameter lambda.
Hence, the limiting distribution of the Negative Binomial distribution is Poisson distribution.

Now let us consider the following result.
If x 1, x2 etc x r is independently, identically distributed random variables having the geometric distribution with parameter p then prove that summation x i follows Negative Binomial distribution.

We prove this result using moment generating function of the distribution.
We know that the moment generating function of geometric distribution is,
M x of t is equal to p into 1 – q e power t whole power minus 1 and
Moment generating function of Negative Binomial distribution is,
M x of t is equal to Q minus P e power t whole power minus r
Substituting for P and Q in terms of p and q, we get
1 by p minus q by p into e power t whole power minus r.
On simplification, we get
P power r into 1 minus q into e power t whole power minus r.

Consider the mgf of summation x i
M summation x i of t is equal to product M x i of t
Is equal to product of p into 1 minus q e power t whole power minus 1

Since same term is multiplied r times we get,
p into 1 minus q e power t whole power minus 1whole power r.
Is equal to p power r into 1 minus q into e power t whole power minus r

which is the mgf of Negative Binomial distribution. Hence by uniqueness theorem of mgf, summation x i has Negative Binomial distribution with parameters r and p.

Now let us prove the additive property of the Negative Binomial distribution.
If x1, x2 etc., x n are independent Negative Binomial variates such that x i follows Negative Binomial distribution with parameters r i and p then show that summation x i follows Negative Binomial distribution with parameters summation r i and p

We prove this using mgf of the distribution.
We know that for Negative Binomial distribution, M x i of t is equal to p power r i into 1 minus q e power t whole power minus r i.

Consider the mgf of summation x i
M summation x i of t is equal product of M x i of t
Is equal to product p power r i into 1 minus q into e power t whole power minus r i
Is equal to p power r 1 into 1 minus q e power t whole power r 1, into p power r 2 into 1 minus q e power t whole power minus r 2, into etc., p power r n into 1 minus q e power t whole power minus r n
Which is same as:
P power summation r i into 1 minus q e power t whole power minus summation r i,

which is the mgf of Negative Binomial distribution. Hence by uniqueness theorem of mgf, summation x i has Negative Binomial distribution with parameters summation r i and p.