1. Introduction

Welcome to the series of E-learning modules on Geometric Distribution. In this module we are going cover the definition of geometric distribution, moments, nature of the distribution, mgf, cgf and result.

By the end of this session, you will be able to explain:

- About Geometric Distribution
- Markov property or Memoryless property
- Raw Moments and Central Moments
- Nature of the distribution
- Moment Generating Function
- Cumulants Generating Function
- Characteristics of Generating Function

Suppose we have a series of independent trials or repetition and on each repetition or trial the probability of success ‘p’ remains constant. Then the probability that there are x failures preceding the first success is given by $q^x p$.

Definition
A random variable x is said to follow a geometric distribution with parameter p, if its probability mass function is given by,

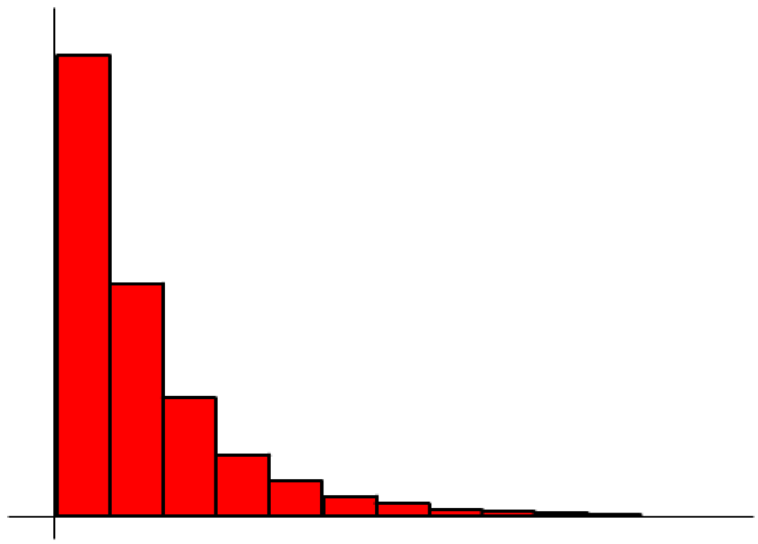
P of x is equal to q power x into p, x take values 0, 1, 2, etc., and p lies between zero and one And equal to zero otherwise.

We write x follows geometric p, read as x follows geometric distribution with parameter p.

Since the various probabilities for x is equal to zero, 1, 2, etc., and are the various terms of geometric progression, hence known as geometric distribution.

The graph of geometric distribution can be drawn as follows:

Figure 1



The assignment of probabilities in p of x is permissible since,
Summation over x from zero to infinity p of X is equal to x
Is equal to summation over x, q power x into p
Is equal to p into one plus q plus q square plus q cube plus etc.
Is equal to p by 1 – q
Is equal to p by p
Equal to one.

Conditions for geometric distribution:

1. An experiment consists of repeating trials until first success.
2. Each trial has two possible outcomes
   - A success with probability p
   - A failure with probability q = 1-p
3. Repeated Trials are independent.

2. Examples of Geometric Distribution

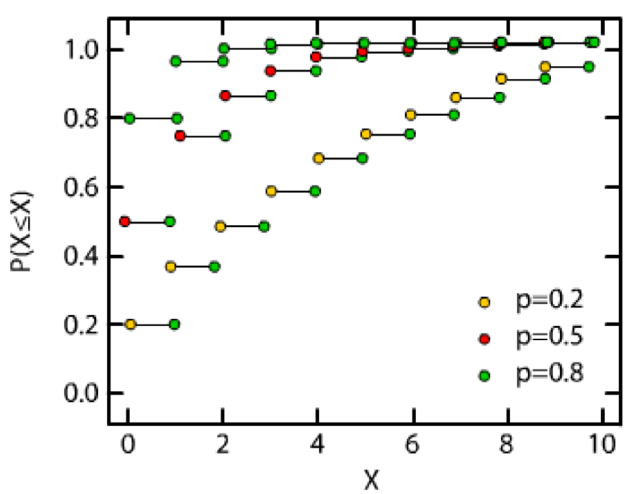
Following are the Examples of geometric distribution:

1. Tossing a coin repeatedly until head occurs.
2. Observing single births until a girl.
3. Roll a die until it results in number 5.
4. Products are inspected until first defective is encountered.
5. If 1 percent of the bits transmitted through a digital transmission are received in error, then bits are transmitted until the first error.
6. If terminals on an on-line computer system are attached to a communication line to the central computer system, then the number of terminals polled the first ready terminal is located.

Cumulative Distribution Function of Geometric Distribution is given by,
F of x is equal to summation from zero to x p into q power x
Is equal to p into 1 plus q plus q square plus q cube plus etc., plus q power x
Is equal to p into 1 minus q power x plus 1 divided by 1 minus q
Is equal to 1 minus q power x plus 1.

The graph of cumulative distribution function for different values of p is as follows:

Figure 2



Markov Property or Lack of memory property of geometric distribution.

The geometric distribution is said to lack memory in a certain sense. Suppose an event E can occur at one of the times t is equal to zero, 1, 2, etc., and the occurrence or waiting time x has geometric distribution. Thus, p of x equal to t is equal to q power t into p where t take values zero 1, 2, etc.

Suppose we know that the event E has not occurred before k, that is, x greater than equal to

k. Let Y = X - k. Thus, Y is the amount of additional time needed for E to occur. We can show that,

Probability of y is equal to t given x greater than or equal to k
Is equal to probability of x is equal to t
Is equal to p into q power t, which implies that the additional time to wait has the same distribution as initial time to wait.

Since the distribution does not depend upon k, it, in a sense 'lacks memory' of how much we shifted the time origin. If 'B' were waiting for the event E and is relieved by 'C' immediately before time k, then the waiting time distribution of 'C' is the same as that of 'B'.

We prove this property as follows.
We have p of x greater than or equal to r
Is equal to summation over s from r to infinity, p into q power s.
Is equal to p into q power r plus q power r plus one plus etc.
Is equal to p into q power r divided by 1 minus q
Is equal to q power r.

Consider the conditional probability
P of y greater than equal to t given x greater than equal k.

Is equal to probability of y greater than equal to t intersection x greater than equal to k divided by probability of x greater than equal to k.

Is equal to probability of x minus k greater than or equal to t intersection x greater than or equal to k divided by probability of x greater than or equal to k

Is equal to x greater than or equal to k plus t divided by probability of x greater than equal to k

Is equal to q power t plus k divided by q power k
Is equal to q power t.

Therefore,
Probability of y is equal to t given x greater than or equal to k
Is equal to probability of y greater than or equal to t given x greater than or equal to k
Minus probability of y greater than or equal to t plus one given x greater than or equal to k
Is equal to q power t minus q power t plus 1
Is equal to q power t into 1 – q
Is equal to q power t into p which is same as probability of x is equal to t.
Hence the proof.

3. Moments of Distribution

Now let us find the moments of the distribution.

Mew one dash is equal to expectation of x
Is equal to summation x into p of x
Is equal to summation over x from zero to infinity x into p q power x
Is equal to p q into summation over x, x into q power x minus 1
Is equal to p q into 1 – q power minus two
Is equal to q by p.

2^{nd} raw moment is given expectation of x square
Is equal to expectation of x into x -1 plus x
Is equal to summation over x, x into x -1 plus x into p of x
Is equal to summation over x from zero to infinity, x into x -1 plus x into p into q power x

Multiplying and dividing by two and adding and subtracting 2 in the power of q and then simplifying, we get,
2 p q square, summation over x from 1 to infinity, x into x-1 divided by 2 into 1, into q power x-2 plus q by p
Is equal to 2 p q square into1 minus q whole power minus 3 plus q by p
By substituting 1 – q is equal to p, we get,
2 q square by p square plus q by p.

Third raw moment is given by,
Mew 3 dash is equal to expectation of x cube
Is equal to expectation of x into x minus 1 into x minus 2 plus 3 x into x minus 1 plus x
Is equal to summation x into x minus 1 into x minus 2 plus 3 x into x minus 1 plus x into p of x
Is equal to summation over x from zero to infinity x into x minus 1 into x minus 2 plus 3 x into x minus 1 plus x into p into q power x

Doing the adjustments as we have done for mew 2 dash, and simplifying, we get,
6 p q cube into summation over x from 2 to infinity, x into x minus 1 into x minus 2 divided by 3 into 2 into 1, into q power x minus 3 plus 6 q square by p square plus q by p
Is equal to 6 p into q cube into 1 minus q power minus 4, plus 6 q square by p square plus q by p
Is equal to q cube by p cube plus 6 q square by p square plus q by p.

Fourth raw moment is given by,
Mew 4 dash is equal to expectation of x power 4
Is equal to expectation of x into x minus 1 into x minus 2 into x minus 3 plus 6 x into x minus 1 into x minus 2 plus 7 x into x minus 1 plus x Is equal to summation x into x minus 1 into x minus 2 into x minus 3 plus 6 x into x minus 1 into x minus 2 plus 7 x into x minus 1 plus x into p of x
Is equal to summation over x from zero to infinity x into x minus 1 into x minus 2 into x minus 3 plus 6 x into x minus 1 into x minus 2 plus 7 x into x minus 1 plus x into p into q power x

Doing the adjustments as we have done for mew 3 dash, and simplifying, we get,
24 p into q power 4 into summation over x from 3 to infinity, x into x minus 1 into x minus 2 into x minus 3 divided by 4 into 3 into 2 into 1, into q power x minus 4, plus 36 q cube by p cube plus 14 q square by p square plus q by p.
Is equal to 24 p into q power four into 1 minus q power minus 5 plus 36 q cube by p cube plus 14 q square by p square plus q by p

Is equal to 24 q power 4 by p power 4 plus 36 q cube by p cube plus 14 q square by p square plus q by p.

4. Central Moments using Raw Moments & Nature of Distribution

Now let us find the central moments using raw moments.

Mew 2 is equal to mew 2 dash minus mew 1 dash whole square
Is equal to 2into q square by p square plus q by p minus q by p the whole square
By simplifying we get,
Q by p square.

Hence, mean of the distribution is q by p and variance is q by p square.
Observe that for geometric distribution, mean is less than variance.

Third central moment is given by,
Mew 3 is equal to mew 3 dash minus 3 into mew two dash into mew 1 dash plus 2 mew one dash cube.
Is equal to 6 q cube by p cube plus 6 q square by p square plus q by p, minus 3 into 2 q square by p square plus q by p into q by p, plus 2 into q by p whole cube.
On simplification we get,
Mew 3 is equal to q into 2 minus p whole divided by p cube.

Mew 4 is equal to mew 4 dash, minus 4 into mew 3 dash into mew 1 dash, plus 6 into mew 2 dash into mew 1 dash square, minus three into mew 1 dash power four
Is equal to 24 q power 4 by p power 4 plus 36 q cube by p cube plus 14 q square by p square plus q by p,
Minus 4 into 6 q cube by p cube plus 6 q square by p square plus q by p into q by p,
Plus 6 into 2 q square by p square plus q by p into q by p whole square,
minus 3 into p by p whole power 4.
On simplification, we get,
Q into p square minus 9 p plus 9 whole divided by, p power 4.

Now let us discuss about the nature of the distribution
Coefficient of skewness is given by,
Beta 1 is equal to mew 3 square by mew 2 cube which is equal to 2 minus p whole divided by q
Hence the geometric distribution is positively skewed.

Coefficient of Kurtosis is given by,
Beta 2 is equal to mew 4 by mew 2 square
Is equal to 9 plus p square by q.

Since beta 2 is greater than 3, the distribution has leptokurtic curve.

Now let us find the moment generating function.
M x of t is equal to expectation of e power t x

Is equal to summation over x from zero to infinity, e power t x into p of x
Is equal to summation over x, e power t x into p into q power x
Is equal to p into summation over x, e power t into q whole power x
Is equal to p into 1 minus q e power t whole power minus 1
Is equal to p by 1 minus q e power t.

Let us obtain the Cumulants Generating Function of geometric distribution.
Cumulant generating function is obtained by taking logarithm of moment generating function and is denoted by k x of t

That is k x of t is equal to log m x of t
Is equal to log p by 1 minus q e power t
Is equal to log p minus log 1 minus q e power t
By using log expansion we get,
Log p minus, q into e power t minus, minus q e power t whole square divided by 2, minus p e power t whole cube divided by 3, minus q e power t whole power 4 by four etc.

Now let us substitute exponential expansion for e power t and take minus sign outside in the second term to get,
Log p plus q into 1 plus t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 factorial etc., plus q square by 2 into 1 plus t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 factorial etc., plus q cube by 3 into 1 plus t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 factorial etc., plus q power 4 by 4 into 1 plus t plus t square by 2 factorial plus t cube by 3 factorial plus t power 4 factorial etc., plus etc.

By equating the coefficient of t power r by r factorial in K x of t, we get rth Cumulants.

5. Characteristic Generating Function of Geometric Distribution

Characteristic Generating Function of geometric distribution is given by,
phi x of t is equal to expectation of e power i t x
Is equal to summation over x from zero to infinity, e power i t x into p of x
Is equal to summation over x, e power i t x into p into q power x
Is equal to p into summation over x, e power i t into q whole power x
Is equal to p into 1 minus q e power i t whole power minus 1
Is equal to p by 1 minus q e power i t.

Let us consider the following result.

Let the two independent random variables X_1 and X_2 have the same geometric distribution. Show that the conditional distribution of X_1 given x 1 plus x 2 is equal to n, is discrete uniform.

We prove this result as follows:
Probability that x 1 is equal to k is equal to probability of x 2 is equal to k is p into q power k, k take values zero, 1, 2 etc.

Consider probability of x 1 is equal to r given x 1 plus x 2 is equal to n
Is equal to probability of x 1 is equal to r intersection x 1 plus x 2 is equal to n divided by probability of x 1 plus x 2 is equal to n.

Is equal to probability of x 1 is equal to r intersection x 2 is equal to n minus r divided by probability of x 1 plus x 2 is equal to n.
Is equal to probability of x 1 is equal to r intersection x 2 is equal to n minus r divided by summation over s from zero to n probability of x 1 is equal to s intersection x 2 is equal to n minus s.

Since x 1 and x 2 are independent, we get,
probability of x 1 is equal to r into probability of x 2 is equal to n minus r divided by summation over s from zero to n probability of x 1 is equal to s into probability of x 2 is equal to n minus s.

Therefore,
probability of x 1 is equal to r given x 1 plus x 2 is equal to n
is equal to p into q power r into p into q power n minus r divided by summation over s from zero to n, to p into q power s into p into q power n minus s.

Is equal to p square into q power n divided by summation over s p square into q power n

Summation over s can be written as n plus one and we get,
p square into q power n divided by n + 1 into p square into q power n
by cancelling common terms in both numerator and denominator, we get,
1 by n + 1, r takes values zero, 1, 2, etc., n

which is the probability mass function of discrete uniform distribution.

Hence the conditional distribution of X one given x one plus x 2 is equal to n is discrete uniform.

Here’s a summary of our learning in this session:

- Definition of Geometric Distribution
- Markov property or Memoryless property
- Raw Moments and Central Moments
- Nature of the distribution
- Moment Generating Function
- Cumulants Generating Function
- Characteristic Generating Function