1. Introduction and Definition of Probability

Welcome to the series of E-learning modules on Definition of Probability, Classical and Relative frequency approach to probability. In this module we are going cover the basic concept of probability, definitions of Classical and Relative Frequency Approach, Limitations of each definition of probability.

By the end of this session, you will be able to know:

- The basic concepts of probability
- Classical Approach and its limitations
- Relative frequency Approach and its limitations
- Comparison between the above two

The word Probability or Chance is very commonly used in day to day conversation and generally people have a vague idea about its meaning. For e.g.: We come across statements like “It may rain tomorrow”, “The chances of teams A and B winning a certain match are equal”, etc.

All these terms mostly, probably, presumably, likely etc. convey the same meaning, that is, the event is not certain to take place or in other words there is an uncertainty about happening of the events in question. In Statistics we try to present conditions under which we can make sensible numerical statements about uncertainty.

Business decisions are often based on an analysis of uncertainties such as the following

1) What are the chances that sales will decrease if we increase the prices?
2) What is the likelihood that a new assembly method will increase productivity?

Definition of Probability:
Probability is a numerical measure of the likelihood that an event will occur. Thus probability could be used to measure the degree of uncertainty associated with the events listed above.

The probability of a given event is an expression of likelihood or chance of occurrence of an event. There is no general agreement about its interpretation. However, broadly speaking, there are different schools of thought on the concept of probability.

Probability values are always assigned on a scale between zero and one. A probability near zero indicates that an event is unlikely to occur; a probability near one indicates that an event is almost sure to occur. Other probabilities between zero and one represent varying degrees of likelihood that an event will occur. Hence probability is nothing but a numerical measure associated with the occurrence or non-occurrence of an event.

Probability is important in decision making because it provides the way to measure, express and analyze the uncertainties associated with future events

2. Experimental Outcomes and Basic Concepts of Probability

Assigning probabilities to Experimental outcomes:
With an understanding of an experiment and a sample space, let us now see how probabilities for the outcomes can be determined. In assigning probabilities to experimental outcomes two basic requirements of probability must be satisfied.

The first requirement is:
The probability values assigned to each experimental outcome must be between zero and one. If we let E_i denote the i^{th} experimental outcome and P of E_i indicate the probability of this experimental outcome, we must have zero less than or equal to P of E_i less than or equal to one for all i. Call this as (1)

The second requirement is:
The sum of all of the experimental outcome probabilities must be one. For example if a sample space has k experimental outcomes, we must have P of E_1 plus P of E_2 plus P of E_3………till P of E_k is equal to 1. Call this as (2)

Any method of assigning probability values to the experimental outcomes that satisfies these two requirements and results in reasonable numerical measures of the likelihood of outcomes is acceptable. In practice the Classical Approach, the relative frequency Approach and Axiomatic Approach are often used.

Let us just recollect some of the basic concepts of probability which are required for the classical and relative frequency approach of the probability

a) Exhaustive Cases

The total number of possible outcomes of a random experiment is called exhaustive cases for the experiment

Example: In a toss of a single coin there are two exhaustive cases { Head and Tail }

If two coins are tossed the exhaustive cases are 2 square equals to 4. That is,
{ Head Head, Head Tail, Tail Tail, Tail Head }
Therefore for n coins it is two to the power n

ii) Exhaustive cases for a toss of a single die equals to six
Exhaustive cases for a toss of two dice equals to six square
Exhaustive cases for a toss of three dice equals to six to the power three , which equals two hundred and sixteen

iii) In drawing two cards from a pack of cards the exhaustive number of cases is fifty-two C two

Equally Likely cases

The outcomes of a random experiment is said to be equally likely if after taking into consideration all relevant evidence, none of them can be expected in preference to another.

For Example: i)If a coin is unbiased the two outcomes head and tail are equally likely

ii)Drawing of 2 cards from a pack of fifty-two cards , the outcomes are equally likely

Mutually exclusive cases

Events are said to be mutually exclusive when two or more of them cannot occur simultaneously.

For example: In tossing of a coin events Head or Tail are mutually exclusive

3. Classical Approach

Classical Approach:

The classical approach to the probability is the oldest and simplest. It originated in 18th century in the problems pertaining to games of chance, such as throwing of coins, dice or deck of cards etc.

To illustrate the classical approach of assigning probabilities, let us consider the experiment of flipping a coin.

An event, whose probability is sought, consists of one or more possible outcomes of the given activity. On any line flip, we will observe one of the two experimental outcomes: head or tail.

A reasonable assumption is that the two possible outcomes are equally likely. Therefore as one of the two equally likely outcomes is a head, we logically should conclude that the probability of observing a head is half. Similarly, that the probability of observing a tail is half.

When the assumption of equally likely outcomes is used as a basis for assigning probabilities, the approach is referred to as a classical approach. If an experiment has n possible outcomes the classical method would assign a probability of one by n to each of the experimental outcome.

As another illustration of the classical approach, consider the experiment of rolling a die. When a die is rolled once any one of the 6 possible outcomes one, two, three, four, five and six can occur. We can describe a sample space and sample points for this experiment with the notation

S is equal to (within brackets) {1,2,3,4,5,6}

A die is designed so that the six experimental outcomes are equally likely and hence each outcome is assigned a probability of 1 by 6. Thus if P of 1 denotes the probability that one dot appears on the upward face of the die, and then P of 1 equals to 1 by 6. Similarly P of 2 equals to 1by 6 and so on P of 6 equals to 1 by6.

Note that the probability assignment satisfies the two basic requirements of assigning probabilities. In fact requirements (1) and (2) are automatically satisfied when the classical approach is used because each of the n sample points is assigned a probability of 1 by n.

The definition of probability given by a French mathematician Laplace and generally adopted by disciples of the classical school runs as follows:

Probability is the ratio of number of favourable cases to the total number of cases. If the probability of occurrence of A is denoted by P of A then by this definition we have

Probability of A= Number of favourable cases by Total number of cases

For calculating probability we have to find out two things

Number of favourable cases and,

Exhaustive number of cases

Symbolically , if an event A can happen in ‘m’ ways out of a total of ‘n’ equally likely and mutually exclusive ways then the probability of the occurrence of the event is denoted by p is equal to Probability of A equals to m by n

And the probability of the non-occurrence of the event is given by q equals to P of A complement, equal to n minus m by n equals to m dash by n equals to 1 minus m by n equals to 1 minus P of A equals to 1 minus p
Since the sum of the successful and unsuccessful outcomes is equal to the total number of events we have
m plus m’ dash is equal to n
Dividing by n
m by n + m’ dash by n equal to 1
so that p plus q equals to 1

Probability therefore may be written as a ratio. The numerator of the fraction corresponding to this ratio represents the number of successful (or unsuccessful) outcomes while the denominator represents the total number of possible outcomes.

Classical probability is often called a priori probability because if we keep using orderly examples of unbiased dice, fair coin, etc., we can state the answer in advance (a priori) without rolling a dice, tossing a coin etc.

4. Illustration of Classical Approach and Introduction to Relative Frequency Approach

Illustration:
From a bag containing 10 black and 20 white balls a ball is drawn at random. What is the probability that it is white?

Total number of balls in the bag equals to 30
Number of white balls equals to 20
Probability of getting a white ball or
p equal to P of A equals to Number of favourable cases divided by Total number of equally likely cases equals to 20 by 30

Probability of not getting a white ball or q equals to 10 by 30
Thus p plus q equals to 2 by 3 plus 1 by 3 equal to 1

A classical approach was developed originally to analyze the gambling probabilities where the assumption of equally likely outcomes often is reasonable. In many business problems however this assumption is not valid.

Limitations of Classical Approach:
The classical definition of probability given above suffers from certain limitations. The definition cannot be applied whenever it is not possible to make a simple enumeration of equally likely and mutually exclusive cases.
For example: How does it apply to probability of rain? We might think that there is two possibilities 'rain' or 'no rain'. But at any given time it will not usually be agreed that they are equally likely

The phrase "equally likely" appearing in the classical definition is synonymous with "equally probable". How do we know whether the probability is equal before we can measure them?
If a person jumps from the top of Qutab Minar the probability of his survival will not be 50 percent since survival and death, the two mutually exclusive outcomes are not equally likely.

The definition has only limited applications in coin tossing, die throwing and similar games of chance. The classical approach also fails to answer the questions like

i) What is the probability that a male will die before the age 60?

Or, a bulb will burn less than two thousand hours? Etc.
The definition fails when the number of possible outcomes is infinitely large.

Real life situations unlikely and disorderly as they often are make it difficult and at times impossible to apply classical probability concept.

Relative Frequency Approach:

In the one thousand eight hundred's British statisticians interested in theoretical foundations for calculating risk of losses in life insurance and commercial insurance, began defining probabilities from a statistical data collected on births and deaths. Today, this approach is called relative frequency of occurrence.

The classical approach is difficult or impossible to apply as soon as we deviate from the fields of coins, dice, cards and other simple games of chance. Secondly the classical approach may not explain actual results in certain cases.
For example: if a coin is tossed 10 times, we may get 6 heads and 4 tails. The probability of head is thus 0 point 6 and that of a tail is 0 point 4.

However if the experiment is carried out a larger number of times we should expect approximately equal number of heads and tails. As n increases, that are approaches to infinity, we find that the probability of getting a head or tail approaches 0 point5. The probability of an event can thus be defined as the relative frequency with which it occurs in n, that is, indefinitely large number of trials.

Consider a firm that is preparing to a market a new product. In order to estimate the probability that a customer will purchase the product, a test market evaluation has been set up wherein sales people call on potential customers.

Each sales call conducted has two possible outcomes: The customer purchases the product or the customer does not purchase the product. With no reasons to assume that the two experimental outcomes are equally likely the classical method of assigning probabilities is inappropriate.

5. Relative Frequency Approach Contd

Suppose that in the test market evaluation of the product, four hundred potential customers were contacted, hundred purchased the product, but three hundred did not. In fact we have repeated the experiment of contacting the customers four hundred times and found that the product was purchased hundred times.
Thus we might decide to use the relative frequency of number of customers that purchased the product as an estimate of the probability of a customer making a purchase. We could assign a probability of hundred by four hundred equal to point two five to the experimental outcome of purchasing the product. This approach to assigning probabilities is referred to as the relative frequency method.

If an event occurs m times out of n, its relative frequency is m by n, that is, the value which is approached by m by n when n becomes infinity is called the limit of the relative frequency.
Symbolically
P(A)= Limit of n tends to infinity m by n
Theoretically we can never obtain the probability of an event as given by the above limit. However in practice we can try to have a close estimate of P of A based on large number of observations that is n.

In the relative frequency definition the fact that the probability is the value which is approached by m/n when n becomes infinity, emphasizes a very important point, that is, probability involves the long term concept. This means that if we toss a coin only 10 times we may not get exactly 5 heads and 5 tails. However, if an experiment is carried out larger and larger number of times, say a coin is thrown ten thousand times , we can expect the outcome of heads and tails to be very close to fifty percent

Comparison of the two definitions:
The two approaches classical and relative frequency or empirical though seemingly same, differ widely. In the former, P of A and m by n were practically equal when n was large whereas, in the latter we say that P of A is the limit m by n as n tends to infinity. In the second approach thus the probability itself is the limit of the relative frequency as the number of observations increases indefinitely.
The probability obtained by following relative frequency definition is called a posterior or empirical probability as distinguished from a priori probability obtained by the classical approach.

Priori and Posterior probability:
A clear distinction between a priori and empirical probability is quite important for the proper understanding of the concept of probability
Priori probability is normally encountered in problems dealing with games of chance- for example: Dice and cards.
Normally we think of a priori probability as being deductive in nature i.e., from cause to effect and based on theory instead of evidence of experience and experimentation.

Probability derived from the past experience is called an empirical probability and is used in many practical problems. The classic example being the preparation of Insurance mortality tables which obviously are based on past experience. In the analysis of most of the practical business problems these two kinds of probability concepts are widely used.

A priori probability may be determined rather easily whenever a complete enumeration is available on the various ways an event may occur
For example: the probability of drawing a king from a deck of fifty two cards is 4 by fifty two or 1 by thirteen
Similarly the probability of getting 3 in a single throw of a dice is 1 by 6

Limitations of Relative Frequency Approach:
The relative frequency approach though useful in practice has difficulties from the mathematical point of view
1) An actual limiting number may not really exist. Quite often people use this approach without evaluating a sufficient number of outcomes. For example: Mr Kohli pointed out that his father and mother (aged seventy-five and seventy) both had a serious heart problem in Jan-Feb nineteen ninety-nine and hence in winter people above seventy have high probability of heart attack. His friends took it seriously and started to give special attention to their parents.

2) On deep thinking we may find that there is not enough evidence of establishing a relative frequency of occurrence of probability.
3) It may be observed that the empherical probability can never be obtained and one can only attempt at a close estimate of P of A by taking n sufficiently large.
For this reason modern probability theory has been developed axiomatically in which probability is an undefined concept much the same as point and line are undefined in Geometry.

Here's a summary of our learning in this session:
- Discussed about the basic notion of probability
- Definitions of probability
- Classical and Relative frequency approach
- Their limitations and comparison between the two