1. Introduction

Welcome to the series of E-learning modules on conditional probability and independent events. In this module we are going to cover the conditional probability of dependent and independent events and also the application of multiplication theorem of probability in real life.

By the end of this session, you will be able to know:

- Meaning of the conditional probability
- Conditional probability of independent events
- Multiplication theorem of probability
- Application of conditional probability and multiplication theorem

Conditional probability is the probability of an event in the sub – population or in the reduced sample space. It is the probability of an event given that the other event is already taken place.

For instance, if we consider any two events A and B then the conditional probability of event B given A is the probability of happening of the event B when the event A has already happened.

Probability of B/A is not defined if P(A)=0 and (A/B) is not defined if P(B) =0 because we cannot divide a number by zero.

<u>Result</u>

For a fixed B with P(B) > 0, P(A/B) is a probability function.

This result can be proved as follows:

To prove this first we should show it satisfies three axioms of probability:

Probability of the event is greater than or equal to zero

P(S) = 1, S is the sample space, and

If { A_n } is any finite or infinite sequence of disjoint events, then

$$P(\bigotimes_{n} A_i) = \sum_{i=1}^{n} P(A_i)$$ **(Formula 1)**

(i.e., probability of union of A_i is same as the summation of Probabilities of A_i)

Now we apply these axioms to the conditional probability:

$$\boldsymbol{P(A/B) = \frac{P(A \cap B)}{P(B)}} \geq 0$$, **(Formula 2)**

since P (A∩B) ≥ 0 and P (B) ≥ 0, as both are probabilities which are greater than or equal to zero and less than one. Also the numerator is smaller than the denominator.

Considering left hand side and replacing A by S in the second condition mentioned above, we get,

$$\boldsymbol{P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1}$$, **(Formula 3)**

(since S is a sample space and B is a subset, their intersection will be set B.)

If { A_n } is any finite or infinite sequence of disjoint events, then

$$P(\bigotimes_n A_n / B) = \frac{P(\bigotimes_n A_n \cap B)}{P(B)} \quad \textbf{(Formula 4)}$$

(probability of union of A_n given B is equal to the probability of union of A_n intersection B divided by P(B))

$$= \frac{P(\bigotimes_n A_n . B)}{P(B)}$$

$$= \frac{\sum_n P(A_n . B)}{P(B)}$$

$$= \sum_n \frac{P(A_n . B)}{P(B)}$$, since the denominator is independent of n we can take it inside the summation.

$$= \sum_n P(A_n / B) \quad \textbf{(Formula 5)}$$

Here we proved that P (A/B) satisfies all the axioms of the probability.
Therefore P (A/B) is a probability function.

2. Independent Events

Independent events: Two events are said to be independent if occurrence or non–occurrence of one does not depend on the occurrence or non–occurrence of the other.

Conditional probability of independent events:

If the two events are independent then conditional probability and unconditional probabilities become equal.

Therefore, $P(A/B)= P(A)$ and $P(B/A)= P(B)$.

That is outcome of one event does not influence the other event.

Result 1 - If the events A and B are such that $P(A) \neq 0$, $P(B) \neq 0$ and A is independent of B, then B is independent of A.

This result is proved as follows:
Given that A is independent of B.
Therefore $P(A/B)=P(A)$
Implies $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ **(Formula 6)**
Implies $P(A \cap B) = P(A)\,P(B)$
Implies $\dfrac{P(B \cap A)}{P(A)} = P(B)$, **(Formula 7)**
since $P(A \cap B) = P(B \cap A)$ and $P(A) \neq 0$
Implies $P(B/A) = P(B)$
Which by definition of independent events, means that B is independent of A
Result - 2

If A and B are independent events, then

i. A and B' are independent
ii. A' and B are independent
iii. A' and B' are independent

Where A' is complementary event of A and B' is the complementary event of B.

Above result can be proved as follows:

(i) If A and B are two independent events then the probability of simultaneous occurrence of A and B is given by:

$P(A/B) = P(A)$ and $P(B/A) = P(B)$

Also $P(A \cap B) = P(A).P(B) = P(A/B).P(B) = P(A) . P(B/A)$

As B and B' are complementary events

$P(B) + P(B') = 1$

Consider $P(B/A)+ P(B'/A) = 1$

$P(B) + P(B'/A) =1$ (substituting $P(B/A) = P(B)$)

Implies $P(B'/A) = 1 - P(B) = P(B')$

Hence A and B’ are independent.

(ii) As A and A’ are complementary events

$P(A) + P(A') = 1$

Consider $P(A/B) + P(A'/B) = 1$

$P(A) + P(A'/B) = 1$ (substituting $P(A/B) = P(A)$)

Implies $P(A'/B) = 1 - P(A) = P(A')$

Hence A and B’ are independent.

(iii) From (i) we have A and B’ are independent

Therefore $P(A/B') = P(A)$ and $P(B'/A) = P(B')$

Therefore, $P(A/B') + P(A'/B') = 1$

$P(A) + P(A'/B') = 1$ (Substituting $P(A/B') = P(A)$)

Implies $P(A'/B') = 1 - P(A) = P(A')$

Hence A’ and B’ are independent.

3. Multiplication Theorem of Probability

From the definition of conditional probability we can define multiplication theorem of probability, which is the probability of happening of two or more events simultaneously.
$P(A \cap B) = P(A).P(B/A)$

If the events are independent, the statement reduces to
$P(A \cap B) = P(A).P(B)$

<u>Result:</u> In the above slide we have given the multiplication theorem for two events. Suppose if there are n events, namely, $A_1, A_2, \ldots, A_n$ then the probability of happening of n events simultaneously is given by:
$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1).P(A_2 /A_1).P(A_3 /A_1 \cap A_2) \ldots P(A_n /A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

This result can be proved using mathematical induction.
If there are two events A_1 and A_2, then
$P(A_1 \cap A_2) = P(A_1). P(A_2/A_1)$, the probability of event A_1 and the Probability of event A_2 given that the event A_1 is already happened.

If there are three events A_1 A_2 and A_3, then
$P(A_1 \cap A_2 \cap A_3) = P(A_1). P(A_2/A_1).P(A_3/A_1 \cap A_2)$, the probability of event A_1, the Probability of event A_2 given that the event A_1 is already happened and the probability of event A_3 given that the events A_1 and A_2 are already happened.

Similarly if there are k events then,
$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1).P(A_2 /A_1).P(A_3 /A_1 \cap A_2) \ldots P(A_k /A_1 \cap A_2 \cap \ldots \cap A_{k-1})$,
the probability of event A_1, the Probability of event A_2 given that the event A_1 is already happened, the probability of event A_3 given that the events A_1 and A_2 are already happened and proceeding like this, the last term will be the probability of the event A_k given that the events $A_1, A_2, \ldots, A_{K-1}$ are already happened.

Therefore in general if there are n events then,

$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1).P(A_2 /A_1).P(A_3 /A_1 \cap A_2) \ldots P(A_n /A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

4. Application of Conditional Probability

Illustration - 1

Here is a problem of drawing a card from a pack of cards which is a heart given that it belongs to red suit.

In a pack of cards there are 52 cards and 26 of re suit and 26 of black suit. Further there are 13 cards which belong to hearts.
Let us define the events
A: Card drawn is red and
B: Card drawn is a heart.
By definition of probability,
P (A) =26/52, P (B) =13/52.
Further happening of both the events,
P (A∩B) =13/52, Cards belong to red suit and heart.

Therefore by definition of conditional probability, the probability of drawing a heart given that it is red is given by:

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{{}^{13}\!/_{52}}{{}^{26}\!/_{52}} = \frac{1}{2}$$

(Formula 8)

Illustration - 2

Here is another example of throwing two dice, where sum of the numbers obtained on the two dice are noted and we need to find the probability that the numbers obtained on both the dice are even given that the sum is 4.

When we throw two dice, the numbers we can have on both the dice are,
(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)

Therefore we have 36 equally likely, mutually exclusive and exhaustive outcomes while throwing to dice.

We can define the events as follows.
A: The sum of the numbers is 4 and we have three combination of numbers which gives us

the sum 4, ((1, 3), (2, 2), (3, 1)).
Hence P(A) = 3/36

B: The numbers on both the dice are even.
We have only one combination (2, 2), where the sum is 4 and both the numbers are even.
Hence P (A∩B) = 1/36.

Therefore by the definition of conditional probability, the probability of getting even numbers on both the dice given the sum is 4 is given by:

$$P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{1/36}{3/36}=\frac{1}{3}$$ **(Formula 9)**

<u>Illustration -3</u>

Two dice, one green and the other red are thrown. Let A be the event that the sum of the points on the faces shown is odd and B be the event that at least one number1 appears. Find the probability of the events:
(i) A/B (ii) B/A (iii) A'/B' (iv)B'/A'

From illustration 2 we have the sample space of 36 equally likely, mutually exclusive and exhaustive outcomes while throwing to dice (red and green).

Event A is given to be the sum of the points on the faces shown is odd
There are18 combinations where we get the sum odd.
Hence P(A)=18/36 =1/2 and P(A') = 1-P(A) = 1- (1/2) =1/2

Event B is an event of at least one number is 1
That is, {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1),(3,1), (4,1), (5,1), (6,1)}
Observe that there are 11 combination of numbers where we get at least one (one or two) 1 in the combination.
Hence P (B) = 11/36 and P(B') = 1- (11/36) = 25/36

Further the event A∩B has the following outcomes
{(1,2), (1,4), (1,6), (2,1), (4,1), (6,1)}. Observe that there are 6 combinations of numbers has the events both A and B
Hence P(A∩B)= 6/36 = 1/6

A'∩B' is given by using the relation
A'∩B'=(AUB)'
The event AUB has altogether 23 elements. Hence P(AUB) = 23/36
P(A'∩B') = P((AUB)')= 1 – P(AUB) = 1 - (23/36) = 13/36

Having probabilities of different events, We can find:

(i) $$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{1/6}{11/36}=\frac{6}{11}$$ **(Formula 10)**

(ii) $$P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{1/6}{1/2}=\frac{1}{13}$$ **(Formula 11)**

(iii) $$P(A'/B')=\frac{P(A'\cap B')}{P(B')}=\frac{13/36}{25/36}=\frac{13}{25}$$ **(Formula 12)**

(iv) $$P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{13/36}{1/2}=\frac{13}{18}$$ **(Formula 13)**

5. Illustrations and Summary

Illustration - 7

A bag contains 10 gold and 8 silver coins. Two successive drawing of 4 coins are made such that,

Coins are replaced before the 2nd trail

Coins are not replaced before the 2nd trial

Find the probability that the first drawing will give 4 gold and the second drawing will give 4 silver coins.

Let us define two events as follows.
A : Drawing 4 gold coins
B : Drawing 4 silver coins

A bag contains 10 gold coins and 8 silver coins. Hence altogether there are 18 coins and first we draw 4 gold coins. Hence the probability of drawing 4 gold coins will be given by

$$P(A) = \frac{{}^{10}c_4}{{}^{18}c_4} = \frac{10\times9\times8\times7 / 4\times3\times2\times1}{18\times17\times16\times15 / 4\times3\times2\times1} = \frac{7}{102} \quad \textbf{(Formula 14)}$$

When the coins are replaced before the second trial the events A and B become two independent events and hence

P (A∩B) = P (A). P(B)

The probability of selecting 4 silver coins are given by

$$P(B) = \frac{{}^{8}c_4}{{}^{18}c_4} = \frac{8\times7\times6\times5 / 4\times3\times2\times1}{18\times17\times16\times15 / 4\times3\times2\times1} = \frac{7}{306} \quad \textbf{(Formula 15)}$$

Therefore the probability that the first drawing will give 4 gold coins and second draw will give 4 silver coins is given by,

$$P(A \cap B) = \frac{7}{102} \times \frac{7}{306} = \frac{49}{31212} \quad \textbf{(Formula 16)}$$

When the coins are not replaced before the second trial, the event B will depend on the event A and hence

P (A∩B) = P (A). P(B/A)

The probability of selecting 4 silver coins in the second draw is given by,

$$P(B/A) = \frac{{}^{8}c_4}{{}^{14}c_4} = \frac{8\times7\times6\times5 / 4\times3\times2\times1}{14\times13\times12\times11 / 4\times3\times2\times1} = \frac{10}{143}$$ **(Formula 17)**

Therefore the probability that the first drawing will give 4 gold coins and second draw will give 4 silver coins is given by,

P (A∩B) = (7/102)*(10/143)=70/14586

<u>Illustration – 8</u>

A fair coin is tossed thrice. What is the probability that all the three tosses result in heads?

Here we are tossing a fair coin thrice. Since the coin has only two faces head and tail, the probability of getting a head in a toss is 0.5.

Here we can define three events A, B and C as follows since we have toss thrice.

A: The first toss results in head.
B: The second toss results in head.
C: The third toss results in head.

Probability of above events A, B and C are same and equal to 0.5.

Since A, B and C are results of three different tosses, they are independent. Therefore the probability that the number of tosses result in head is given by,
P(A∩B∩C)= 0.5X0.5X0.5=0.125

<u>Illustration 9</u>

The odds against Manager X settling the wage dispute with the workers are 8:6 and odds in favour of manager Y settling the same dispute are 14:16.
What is the chance that neither settles the dispute, if they both try independently of each other?

For this problem let us define the events as follows:
A: The Manager X will settle the dispute.
B: The Manager Y will settle the dispute.

Given that odds against Manager X settling the dispute with the workers are 8:6.
Hence P(A') = 8/(8+6) = 4/7 (odds against – complementary event)
and hence P(A) = 1- 4/7 = 3/7

The odds in favour of Manager Y settling the same dispute is 14:16
Hence P(B)= 14/(14+16) = 7/15 (odds favour – given event)
and hence P(B') = 1 – 7/15 = 8/15

Therefore the required probability that neither settles the dispute is given by

$P(A' \cap B') = P(A') . P(B')$ (using the result when A and B are independent then A' and B' are also independent)

$= 4/7 \times 8/15 = 32/105$

Here's a summary of our learning in this session:

- Meaning of conditional probability.
- Independence of two events and their probability and some basic results.
- Multiplication theorem of probability and results.
- Application of conditional probability and multiplication theorem through different illustrations.