1. Introduction

Welcome to the series of E-learning modules on Moments. In this module we are going to cover the definition, measures, calculation and purpose of moments.

By the end of this session, you will be able to:

- Explain the concept of Moments
- Explain the measures of Moments
- Explain the calculation of Moments
- Explain the purpose of Moments

Introduction
The word ‘Moment’ is a very commonly used mechanical term. Moment refers to a measure of force with respect to its tendency to provide rotation.

The strength of the tendency depends on the amount of force and the distance from the origin of the point at which the force is exerted.

The word Moment in statistics is used to describe the characteristics of a frequency distribution like central tendency, variation, skewness and kurtosis.

The analogy between the term moment both in physics and statistics is that we use it to describe the moment of an item or a random variable at some point.

The statistical definition of the term moment is as follows:

When the symbol ‘x’ is used to represent the deviation of any item in the distribution from the arithmetic mean of that distribution. The arithmetic mean of the various powers of these deviations in any distribution is called the moment of the distribution.

The measure moment in statistics is used to indicate the peculiarities of a frequency distribution. The utility of the moment lies in the sense that they indicate the different aspects of a given distribution. Thus, by using moments we can measure the central tendency of a series, dispersion or variability, skew and peak of the curve.

2. Central Moments for an Ungrouped Data

The moments about the actual arithmetic mean are denoted by the Greek letter mu (μ)
The first four moments about mean or central moments for an ungrouped data are as follows:

The first moment about mean is mu one (μ_1) which is calculated by taking the total of the deviation of the variables from mean and dividing it by the total number of variables. It is symbolically represented as mu one is equal to sigma x minus mean divided by N.

The first moment about the mean is always zero as the sum of the deviations of items from the arithmetic mean is always zero. Therefore, mu one is equal to zero.

The second moment about mean is mu two (μ_2) which is calculated by taking the square of the total of the deviation of the variables from mean and dividing it by the total number of variables. It is symbolically represented as: mu two is equal to sigma x minus mean square divided by n.

The second moment about the mean is always equal to the standard deviation square indicating the variance. Therefore, mu two is equal to standard deviation square.

The third moment about the mean is mu three (μ_3) which is calculated by taking the cube of the total of the deviation of the variables from mean and dividing it by the total number of variables. It is symbolically represented as:
Mu three is equal to sigma x minus mean cube divided by N.

The third moment about the mean is always used to measure the skewness of the data. Skewness means the distribution of the data. Therefore, mu three is equal to skewness.

The fourth moment about the mean is mu four (μ_4) which is calculated by taking to the power of four the total of the deviation of the variables from mean and dividing it by the total number of variables. It is symbolically represented as: Mu four is equal to sigma x minus mean to the power of four divided by N.

The fourth moment about the mean is always used to measure the kurtosis of the data. Kurtosis means the Peakness of the data. Therefore, mu four is equal to kurtosis.

In the case of a grouped data the raw data is grouped and the repeating values are taken as frequencies then the first four moments about mean or central moments for a grouped data are as follows:

The first moment about mean is mu one (μ_1) which is calculated by taking the total of the deviation of the variables from mean multiplying it by frequency and dividing it by the total number of variables. It is symbolically represented as:
Mu one is equal to sigma of frequency into x minus mean divided by N.

The second moment about mean is mu two (μ2) which is calculated by taking the square of the total of the deviation of the variables from mean, multiplying it by the frequency and dividing it by the total number of variables.
It is symbolically represented as: Mu two is equal to square of sigma of frequency into x minus mean divided by N.

The third moment about the mean is mu three (μ3) which is calculated by taking the cube of the total of the deviation of the variables from mean, multiplying it by the frequency and dividing it by the total number of variables. It is symbolically represented as: Mu three is equal to cube of sigma of frequency into x minus mean divided by N.

The fourth moment about the mean is mu four (μ4) which is calculated by taking to the power of four the total of the deviation of the variables from mean and dividing it by the total number of variables.
It is symbolically represented as: Mu four is equal to fourth power of sigma of frequency into x minus mean divided by N.

Moments can be extended to higher powers in a similar way, but generally in practice the first four moments meets our requirement.

As pointed out by Yule and Kendall, “moment of high order, though is important are sensitive to sampling fluctuations. The values calculated for moderate number of observations are quite unreliable and hardly give the results for the labour of computation.

3. Calculation of the Moment and the Moment about the Mean

Let us take an example to understand the calculation of the moment and the moment about the mean.

Example:

Find the first, second, third and fourth moments for the set of numbers 2, 3, 4, 5 and 6.

Solution:
To calculate the first moment that is mean we take the total of the variables and divide it by the total number of data. So in this example we find the mean is equal to two plus three plus four plus five plus six divided by five is equal to twenty divided by five is equal to four.

The second moment is equal to two square plus three square plus four square plus five square plus six square divided by five is equal to ninety by five is equal to eighteen.

The third moment is equal to cube of two plus cube of three plus cube of four plus cube of five plus cube of six divided by five is equal to four hundred forty by five is equal to eighty eight.

The fourth moment is equal to the fourth power of two plus the fourth power of three plus the fourth power of four plus the fourth power of five plus the fourth power of six divided by five is equal to two thousand two hundred and seventy four by five is equal to four hundred fifty four point eight.

Using the same data let us calculate the first, second, third and fourth moments about the mean.

The first moment about the mean mu one is calculated by subtracting the first moment from the variable that is, in this case we will find the total of sigma x minus mean xbar, we will get Σ(x-x) by calculating two minus four plus three minus four plus four minus four plus five minus four plus six minus four divided by five is equal to minus two plus minus one plus zero plus one plus two divided by five is equal to zero.

Figure 1

(X)	($\bar{X}$)	($X-\bar{X}$)
2	4	2-4 = -2
3	4	3-4 = -1
4	4	4-4 = 0
5	4	5-4 = 1
6	4	6-4 = 2
N = 5		$\Sigma(X-\bar{X}) = 0$

If you note as mentioned earlier we see that the first moment about the mean is equal to zero.

Let us now calculate the second moment about the mean. Here mu two is equal to the square of total of Sigma x minus mean $\Sigma(x-\bar{x})$ which is calculated by taking two minus four square plus three minus four square plus four minus four square plus five minus four square plus six minus four square divided by five is equal to minus two square plus minus one square plus zero plus one square plus two square divided by five is equal to four plus one plus zero plus one plus four divided by five is equal to ten divided by two is equal to two. This indicates the variance of the data.

Figure 2

(X)	($\bar{X}$)	($X-\bar{X}$)	$(X-\bar{X})^2$
2	4	2-4 = -2	$(-2)^2 = 4$
3	4	3-4 = -1	$(-1)^2 = 1$
4	4	4-4 = 0	$(0)^2 = 0$
5	4	5-4 = 1	$(1)^2 = 1$
6	4	6-4 = 2	$(2)^2 = 4$
N=5			$\Sigma(X-\bar{X})^2 = 10$

The calculation of the third moment about the mean mu three is obtained by taking the cube value of two minus four plus cube of three minus four plus cube of four minus four plus cube of five minus four plus cube of six minus four divided by five is equal to cube of minus two plus cube of minus one plus zero plus cube of one plus cube of two divided by five is equal to minus eight plus minus one plus zero plus one plus eight divided by five is equal to zero. This indicates the skewness of the data.

Figure 3

(X)	($\bar{X}$)	($X-\bar{X}$)	$(X-\bar{X})^3$
2	4	2-4 = -2	$(-2)^3 = -8$
3	4	3-4 = -1	$(-1)^3 = -1$
4	4	4-4 = 0	$(0)^3 = 0$
5	4	5-4 = 1	$(1)^3 = 1$
6	4	6-4 = 2	$(2)^3 = 8$
N=5			$\Sigma(X-\bar{X})^3 = 0$

The calculation of the fourth moment about the mean mu four is obtained by taking the fourth root value of two minus four plus fourth root of three minus four plus fourth root of four minus four plus fourth root of five minus four plus fourth root of six minus four divided by five is equal to fourth root of minus two plus fourth root of minus one plus zero plus fourth root of one plus fourth root of two divided by five is equal to sixteen plus one plus zero plus one plus sixteen divided by five is equal to six point eight. This indicates the kurtosis of the data.

Figure 4

(X)	$(\bar{X})$	$(X-\bar{X})$	$(X-\bar{X})^4$
2	4	2-4 = -2	$(-2)^4 = 16$
3	4	3-4 = -1	$(-1)^4 = 1$
4	4	4-4 = 0	$(0)^4 = 0$
5	4	5-4 = 1	$(1)^4 = 1$
6	4	6-4 = 2	$(2)^4 = 16$
N=5			$\Sigma(X-\bar{X})^3 = 0$

There are two more constants given by Karl Pearson to calculate the measure of skewness and kurtosis based on the second, third and fourth central moments.
It is denoted as beta one and beta two.

Beta one is calculated as the square of mu three divided by the cube of mu two and beta two is calculated as mu four divided by mu two square.
Beta one measures the skewness and beta two measures the kurtosis.

4. Kinds of Distribution

Data’s have two kinds of distribution:
In a symmetrical distribution all odd moments (mu one, mu three) will always be zero. This is so because when the curve is symmetrical the deviation below the mean will be exactly equal to the deviation above the mean, therefore the positive deviation and negative deviation will exactly balance out.

In asymmetrical distribution the odd moments will not be equal to zero. This is so because when the curve is asymmetrical the deviation below the mean will not be exactly equal to the deviation above the mean, therefore the positive deviation and negative deviation will not balance out.

Moments about arbitrary origin:
There are situation when calculating the actual mean where we get the values in fraction then it is difficult to calculate moments by applying the formulae. In such a case we first calculate moments about an arbitrary origin and then convert these moments into moments about the actual mean.

The moments about arbitrary origin is also called the ‘raw moments’ and are denoted by the symbol mu dash (μ’) to distinguish them from the moments about the actual mean.

The four moments about the arbitrary origin ‘A’ is denoted as follows:
mu dash one(μ1’), the first moment , mu dash 2(μ2’), for the second moment,mu dash 3(μ3’) for the third moment and mu dash 4(μ4’) for the fourth moment.

The calculations for the moment about an arbitrary origin shall be done as follows:

Mu one dash is equal to sigma x minus A divided by N where in x is the variable A is the arbitrary mean and N is the total number of the data.
Similarly we calculate for the other moments about an arbitrary origin;
$\mu_2' = \Sigma(x-A)2/N$

Mu two dash is equal to sigma x minus A whole t the power of two divided by N.
$\mu_3' = \Sigma(x-A)3/N$

Mu three dash is equal to sigma x minus A whole t the power of three divided by N.
$\mu_4' = \Sigma(x-A)4/N$

Mu four dash is equal to sigma x minus A whole t the power of four divided by N.

Moments about Zero:
Moments about zero are denoted by (v) and are obtained as follows:

The first moment about zero is equal to arbitrary origin plus mu one dash.$v_1 = A + \mu_2$ or the mean.

The second moment about zero is equal to mu two plus first moment about zero square.
$v_z=\mu_2+v_1^2$

The third moment about zero is equal to mu three plus 3 multiplied by first and second moment about zero minus twice cube of first moment about zero.
$v_3=\mu_3+3v_1v_2-2v_1^3$

The fourth moment about zero is equal to mu four plus 4 into first and third moment about zero minus six into square of first moment about zero into second moment about zero plus three into fourth root of the first moment about zero.
$v_4=\mu_4+4v_1v_3-6v_1^2v_2+3v_1^4$

Sheppard’s correction for grouping Error:

While calculating moments it is assumed that all the values of a variable in a class interval are concentrated at the centre of that interval (that is the mid-point), however in reality the assumption is an approximation for facilitating calculation and it introduces some error which is known as grouping error. The Sheppard's correction is used to eliminate the grouping error.

The moments that we have computed which have not been corrected by Sheppard’s process are called the crude moments and are distinguished from the adjusted moments which we get by applying Sheppard’s corrections.

These corrections are:
Mu one does not need a correction.

mu 2 corrected is equal to uncorrected mu two –the square of the width of the class interval divided by twelve.
$\mu 2$ (corrected) = $\mu 2$(uncorrected) –i2/12

mu three needs no correction

mu four corrected is equal to mu 4 uncorrected minus half width of class interval square into mu two corrected plus seven by two forty into fourth root of the width of the class interval.
$\mu 4$ (corrected) =$\mu 4$ (uncorrected) -1/2 i2$\mu 2$ (uncorrected)+7/240i4.

Conditions for applying Sheppard’s corrections:
The following conditions should be satisfied for the application of Sheppard’s correction:

- The correction should not be made unless the frequency is at least thousand otherwise the moments will be more affected by sampling errors than by grouping errors
- The correction is not applicable to J or U-shaped distributions or even to the skew form

- The observation should be related to a continuous variable
- The frequencies should taper off to zero in both directions that is, the curve should approach the base line gradually and slowly at each end of the distribution
- It needs to be noted that the Sheppard’s correction factor to eliminate the grouping error is best suitable only when the data is continuous data having the characteristics described above and when the original measurements are reasonably precise

5. Purpose of the Moments

Purpose of the moments:
The concept of moments is of great significance in statistical work:

- Moments help in measuring the central tendency of a set of observations, their symmetrical, their variability and the height of the peak of the curve

- As the moments helps is obtaining measures of the various characteristics of a frequency distribution, the calculation of the first four moment about the mean is considered in analyzing the frequency distribution

Let us take an example to understand the moments.
From the following data calculate moments about the
(i)Assumed mean 25, (ii) actual mean and (iii) moments about the zero.

Figure 5

Variable	Mid value (m)	f	d	fd	fd^2	fd^3	fd^4
0-10	5	1	-2	-2	4	-8	16
10-20	15	3	-1	-3	9	-27	81
20-30	25	4	0	0	0	0	0
30-40	35	2	1	2	4	8	16
Total		N=10		-3	9	-27	81

Variance	**0-10**	**10-20**	**20-30**	**30-40**
Frequency	1	3	4	2

Solution:
In this table we will calculate 'd' which is obtained by the mid value minus the arbitrary mean 25 the next column we multiply the frequency with the 'd' value and obtain fd which is the values for the first moment and its square, cube and fourth root in the subsequent columns.

Figure 6

Variable	Mid value (m)	f	d	fd	fd^2	fd^3	fd^4
0-10	5	1	-2	-2	4	-8	16
10-20	15	3	-1	-3	9	-27	81
20-30	25	4	0	0	0	0	0
30-40	35	2	1	2	4	8	16
Total		N=10		-3	9	-27	81

First let us calculate the moments about the arbitrary mean.
mu one dash is equal to sigma of frequency and deviation that is minus three divided by the number of the data 'N' which is 10 into the width of the class interval 'i' that is 10 which is equal to minus three is the first moment.

Similarly the second moment is calculated by taking mu two dash is equal to sigma of square of the frequency and deviation 9 divided by the number of the data 'N' 10 into the square of the width of the class interval 'i' 10 which is equal to ninety.

The third moment is equal to minus 900
The fourth moment is equal to 21,000.

Next let us calculate the moments about the actual mean
As we know the moment of the first order is equal to zero, therefore mu one is equal to zero mu two is equal to mu two dash minus mu one dash square which is equal to ninety minus of minus three square is equal to eighty one.

mu three is equal to mu three dash minus three into mu one dash and mu two dash plus 2 into mu one dash cube is equal to minus 900 minus 3 into minus 3 and 90 plus two into minus three cube which is equal to minus 144

mu four is calculated as mu four dash minus four into mu one dash and mu three dash plus six into mu one dash square into mu two dash minus three into mu one dash to the power of four which is equal to 21,000 minus 10,800 plus 4,860 minus 243 is equal to 14,817.

Now let us calculate the moments about the zero
The first moment about the zero is equal to arbitrary mean plus mu dash one is equal to 25 plus minus three is equal to 22.

The second moment about zero is equal to mu two plus square of the first moment about zero is equal to 81 plus 22 square is equal to 565.

The third moment about zero is equal to mu three plus thrice the moment of first and second about the zero minus twice the cube of first moment about zero is equal to 144 plus 3 into

(22) (565) minus 2 into (22)3 is equal to 15,850.

The fourth moment about zero is equal to mu four plus four into first and third moment about zero minus 6 into square of first moment with the second moment about zero plus three into the fourth moment about zero to the power of four which is equal to 14817plus 1394800 minus 1640760 plus 702768 which is equal to 471625.

Here's a summary of our learning in this session:

The following is the summary of how moments help in analyzing a frequency distribution:

- Moment – What it measures
- First moment about the origin – Mean
- Second moment about the mean – Variance
- Third moment about the mean – Skewness
- Four moment about the mean – Kurtosis