Snippets and Problems

Reduction of physical dimensions

The drag force *F* per unit length on a long smooth cylinder is a function of air speed *U*, density ρ, diameter *D* and viscosity μ. However, instead of having to draw hundreds of graphs portraying its variation with all combinations of these parameters, dimensional analysis tells us that the problem can be reduced to a **single** dimensionless relationship

$$c_D = f\left(\mathrm{Re}\right)$$

where c_D is the drag coefficient and *Re* is the Reynolds number.

In this instance dimensional analysis has reduced the number of relevant variables from 5 to 2 and the experimental data to a single graph of c_D against Re.

Buckingham Pi Theorem

The procedure most commonly used to identify both the number and form of the appropriate non-dimensional parameters is referred to as the Buckingham Pi Theorem. The theorem uses the following definitions:

n = the number of independent variables relevant to the problem

j' = the number of independent dimensions found in the n variables

j = the reduction possible in the number of variables necessary to be considered simultaneously

k = the number of independent **Π** terms that can be identified to describe the problem, k = n - j

Summary of Steps:

1. List and count the n variables involved in the problem.
2. List the dimensions of each variable using {MLTθ}. Count the number of basic dimensions (j') for the list of variables being considered.
3. Find j by initially assuming j = j' and look for j repeating variables which do not form a **Π** product. If not successful, reduce j by 1 and repeat the process.
4. Select j scaling, repeating variables which do not form a **Π** product.
5. Form a **Π** term by adding one additional variable and form a power product. Algebraically find the values of the exponents which make the product dimensionless. Repeat the process with each of the remaining variables.
6. Write the combination of dimensionless pi terms in functional form:

$\mathbf{\Pi}_k = f(\mathbf{\Pi}_1, \mathbf{\Pi}_2, \ldots \mathbf{\Pi}_i)$

Π Theorem Example

Consider the following example for viscous pipe flow. The relevant variables for this problem are summarized as follows:

ΔP = pressure drop   ρ = density   V = velocity   D = diameter

μ = viscosity   ε = roughness   L = length

Seven pipe flow variables: {ΔP   ρ, V, D, μ, ε, L }

dependent   independent

Use of the Buckingham **Π** Theorem proceeds as follows:

1. Number of independent variables: n = 7

2. List the dimensions of each variable (use M L T θ):

variables	ΔP	ρ	V	D	μ	ε	L
dimensions	$ML^{-1}T^{-2}$	ML^{-3}	LT^{-1}	L	$ML^{-1}T^{-1}$	L	L

The number of basic dimensions is j' = 3 (M, L, T).

3. Choose j = 3 with the repeating variables being ρ, V, and D. They do not form a dimensionless **Π** term. No combination of the 3 variables will eliminate the mass dimension in density or the time dimension in velocity.

4. This step is described in step 3. The repeating variables again are ρ, V, and D and j = 3. Therefore, k = n – j = 7 – 3 = 4 independent **Π** terms.

5. Form the **Π** terms:

$$\Pi_1 = \rho^a V^b D^c \mu^{-1} = (ML^{-3})^a (LT^{-1})^b L^c (ML^{-1}T^{-1})^{-1}$$

In order for the **Π** term to have no net dimensions, the sum of the exponents for each dimension must be zero. Therefore, summing the exponents for each dimension, we have:

mass: a - 1 = 0 , a = 1
time: - b + 1 = 0, b = 1
length: -3a + b + c + 1 = 0, c = 3 – 1 – 1 = 1

We therefore have

$$\Pi_1 = \frac{\rho V D}{\mu} = \text{Re} = \text{Reynolds number}$$

Note: Changing the initial exponent for M to 1 (from -1) would result in the reciprocal of the same non-dimensional groups. Thus, some experience is useful in obtaining P terms consistent with existing theory.

Repeating the process with the roughness, ε

$$\Pi_2 = \rho^a V^b D^c \varepsilon^1 = (ML^{-3})^a (LT^{-1})^b L^c (L)^1$$

Solving:

mass:	a = 0 , a = 0
time:	- b = 0, b = 0
length:	-3a + b + c + 1 = 0, c = – 1

Π$_2$ = ε / D Roughness ratio

Repeat the process with the length, L.

$$\Pi_3 = \rho^a V^b D^c L^1 = (ML^{-3})^a (LT^{-1})^b L^c (L)^1$$

Solving:

mass:	a = 0 , a = 0
time:	- b = 0, b = 0
length:	-3a + b + c + 1 = 0, c = – 1

Π $_3$ = L / D length-to-diameter ratio

These three are the independent **Π** terms.

Now obtain the dependent **Π** term by adding ΔP

$$\Pi_4 = \rho^a V^b D^c \Delta P^1 = (ML^{-3})^a (LT^{-1})^b L^c (ML^{-1}T^{-2})^1$$

Solving:

mass:	a + 1 = 0 , a = -1
time:	- b - 2 = 0, b = -2
length:	-3a + b + c - 1 = 0, c = 0

$$\Pi_4 = \frac{\Delta P}{\rho V^2}$$ Pressure coefficient

Application of the Buckingham Pi Theorem to the previous list of variables yields the following non-dimensional combinations:

$$\frac{\Delta P}{\rho V^2} = f\left[\frac{\rho V D}{\mu}, \frac{L}{D}, \frac{\varepsilon}{D}\right]$$

or

$$C_p = f\left[\mathrm{Re}, \overline{L}, \overline{\varepsilon}\right]$$

Thus, a non-dimensional pressure loss coefficient for viscous pipe flow would be expected to be a function of (1) the Reynolds number, (2) a non-dimensional pipe length, and (3) a non-dimensional pipe roughness.

Problem 1: **Convert 400 in^3/day to cm^3/min**

Solution

400 in^3	(2.54 cm)^3	1 day	1 hr
Day	(1 in)^3	24 hr	60 min

= 4.56 cm^3/min

Problem 2: **One hundred pounds of water is flowing through a pipe at the rate of 10.0 ft/s. What is the kinetic energy of this water in (ft)(lb$_f$)?**

Solution

Kinetic energy = K = (1/2) mv^2
Assume that the 100 lb of water means the mass of the water

K =

1	100 lb$_m$	(10 ft)^2	
2		s^2	32.174 (ft)(lb$_m$)/(s^2)(lb$_f$)

= 155 (ft)(lb$_f$)

Problem 3: Use the definition $\tau = \mu \dfrac{dU}{dy}$ to determine the dimensions of viscosity

Solution

From the definition

$$\mu = \frac{\tau}{dU/dy} = \frac{force/area}{velocity/length}$$

Hence

$$\mu = \frac{MLT^{-2}/L^2}{LT^{-1}/L} = ML^{-1}T^{-1}$$

Problem 4: Find the dimensionless form of the solution for the thrust force F_T of a propeller if it depends upon the fluid density ρ, the diameter d, the rotational speed ω, and the relative fluid velocity V.

Solution

$$F_T = f(\rho, d, \omega, V)$$

There are five variables and which results in two dimensionless Π terms. Assume this relation in the form

$$F_T \approx \rho^a d^b \omega^c V^e$$

Inserting the dimensions for each variable

$$MLT^{-2} = \left(\frac{M}{L^3}\right)^a (L)^b \left(\frac{1}{T}\right)^c \left(\frac{L}{T}\right)^e$$

Equating the number of each basic unit

$$1 = a$$
$$1 = -3a + b + e \Rightarrow b = 4 - e$$
$$-2 = -c - e \Rightarrow c = 2 - e$$

Rewriting the assumed functional relation

$$F_T \approx \rho d^{4-e} \omega^{2-e} V^e$$

$$\frac{F_T}{\rho\omega^2 d^4} \approx \left(\frac{V}{\omega d}\right)^e$$

$$\frac{F_T}{\rho\omega^2 d^4} = f\left(\frac{V}{\omega d}\right) \text{ is the standard form}$$

Problem 5: In a furnace, 95% of carbon is converted to carbon dioxide and the remainder to carbon monoxide. By material balance, predict the quantities of gases appearing in the flue gases leaving the furnace.

Solution
Carbon converted to CO2 = 95%
Carbon converted to CO = 5%
Basis is 1 kg of carbon
The combustion equations are

$$C + O_2 = CO_2$$

$$C + \frac{1}{2}O_2 = CO$$

From these equations, 44 kg carbon dioxide is formed by combustion of 12 kg carbon, and 28 kg carbon monoxide is formed by combustion of 12 kg carbon.

Then, the amount of CO2 produced,

$$\frac{(44\ kg\ CO_2)(0.95\ kg\ C\ burned)}{12\ kg\ C\ burned} = 3.48 \text{ kg } CO_2$$

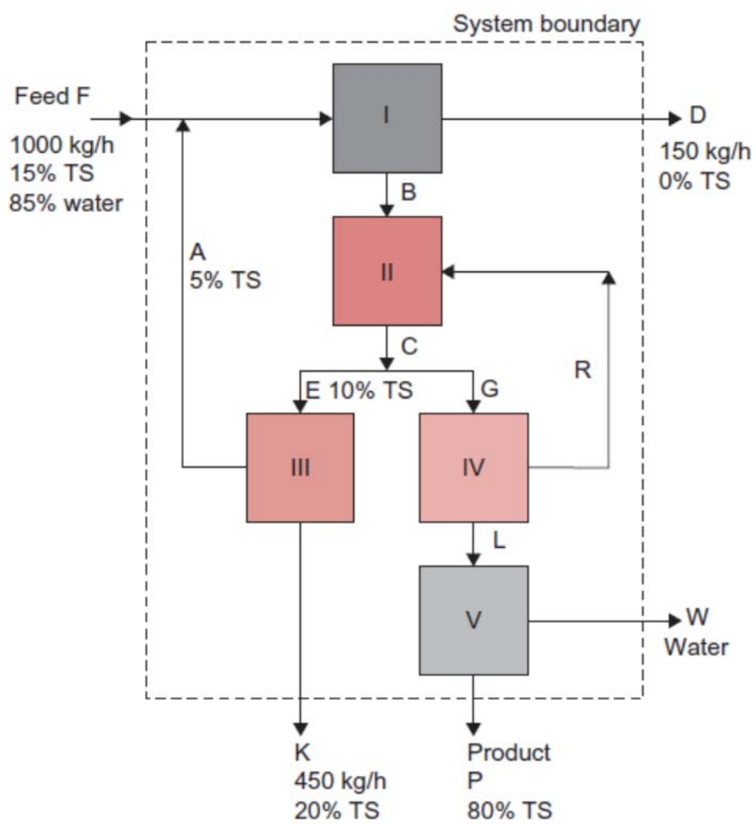
Similarly, the amount of CO produced,

$$\frac{(28\ kg\ CO)(0.05\ kg\ C\ burned)}{12\ kg\ C\ burned} = 0.12 \text{ kg CO}$$

Thus, the flue gases contain 3.48 kg CO2 and 0.12 kg CO for every kilogram of carbon burned.

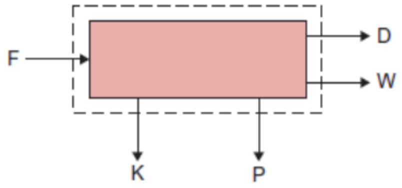
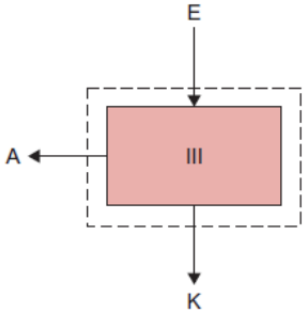
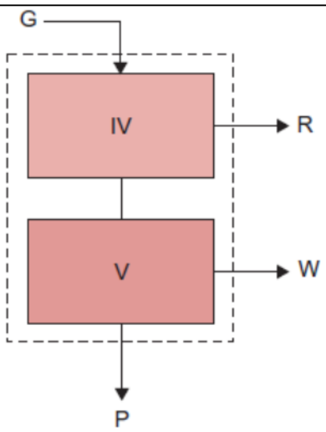
Problem 6: An experimental engineered food is being manufactured using five stages, as shown in Figure below. The feed is 1000 kg/h. Various streams have been labeled along with the known composition values on the diagram. Note that the composition of each stream is in terms of solids and water only. Stream C is divided equally into streams E and G. Product P, with 80% solids, is the desired final product. Stream K produces a by-product at the rate of 450 kg/h with 20% solids. Calculate the following:

a. Calculate the mass flow rate of product P.
b. Calculate the mass flow rate of recycle stream A.
c. Calculate the mass flow rate of recycle stream



Solution

Feed = 1000 kg/h
Solid content of P = 80%
Mass rate of stream K = 450 kg/h
Solids in stream K = 20%

	Basis = 1 h Consider total system, solid balance (See fig. to left) 0.15 x F = (0.2 x K) + (0.8 x P) 0.15 x 1000 = (0.2 x 450) + (0.8 x P) P = 60/0.8 = 75 Kg P= 75 Kg
	Consider stage III (see fig. to left) Total balance E = A + K; E = A + 450 (1) Solid balance 0.1E = 0.05A + 0.2K 0.1E = 0.05A + (0.2 x 450) 0.1E = 0.05A + 90 (2) Solving (1) and (2) simultaneously E = 1350 Kg A = 900 Kg Since C is divided equally into E and G G= 1350 Kg with 10% solid For total system, conduct total balance to find W F = K + P + D + W 1000 = 450 + 75 + 150 + W W = 325 Kg
	Consider stages IV and V together (see fig. to left) G = R + W + P 1350 = R + 325 + 75 R = 950 Kg The mass flow rates of stream P, A, and R are 75 kg, 900 kg, and 950 kg, respectively.