FAQs

(A) Velocity Profile in a Liquid Flowing Under Fully Developed Flow Conditions

Consider fluid flow taking place under steady and fully developed conditions in a constant diameter pipe. Forces due to pressure and gravity cause the fluid flow in the fully developed region. In the case of a horizontal pipe, gravitational effects are negligible. Therefore, for the purpose of this analysis, we will consider only forces due to pressure. When a viscous liquid (a liquid with viscosity greater than zero) flows in a pipe, the viscous forces within the liquid oppose the pressure forces. Application of pressure is therefore necessary for flow to occur, as it overcomes the viscous forces opposing the flow. Furthermore, the flow takes place without accelerating; the velocity profile within the fully developed flow region does not change with location along the x-axis. For the flow to be steady, a balance must exist between the pressure and viscous forces in the liquid.

Using Newton's second law of motion, we can describe forces acting on a small element of liquid as shown in Fig. 1. The cylindrical element is of radius ***r*** and length ***L***. The pipe diameter is ***D***. Initially at time ***t***, the location of the element is identified from **A** to **B**. After a small lapse of time **Δt**, the liquid element moves to a new location **A'B'**. The ends of the element, **A'B'**, indicating velocity profile, are shown distorted, indicating that under fully developed flow conditions, the velocity at the central axis is maximum and decreases with increasing ***r***. Since the pipe is horizontal, we neglect the gravitational forces.

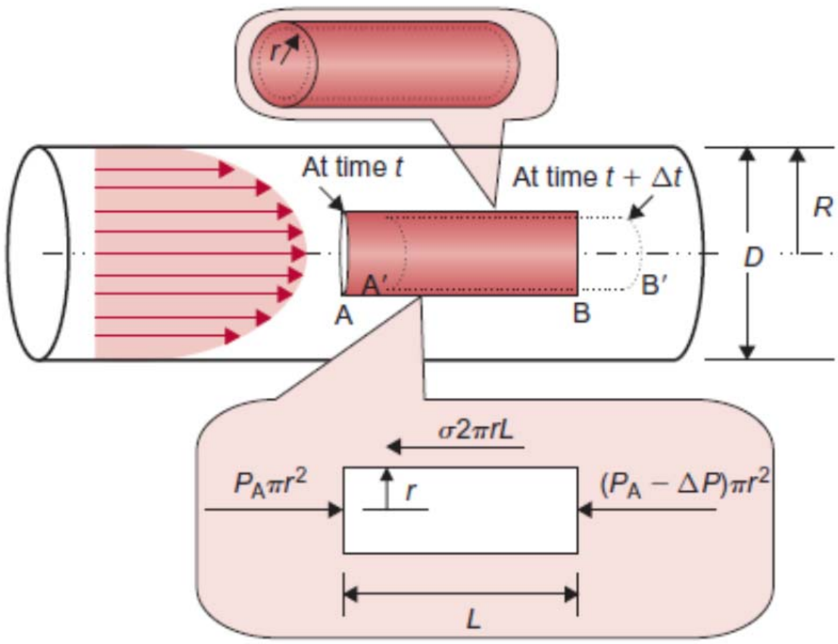


Fig. A: Force balance for a liquid flowing in a pipe

The pressure varies from one axial location to another, but remains constant along any vertical cross-section of the pipe. Let us assume that the pressure on the cross-sectional face at **A** is $\mathbf{P_A}$ and at **B** is $\mathbf{P_B}$. If the decrease of pressure from **A** to **B** is **ΔP**, then $\Delta P = P_A - P_B$

As seen in the force diagram for Fig. A, the pressure forces acting on the liquid element are as follows:

On the vertical cross-sectional area: πr^2
at location A, pressure forces: $P_A \pi r^2$
at location B, pressure forces: $(P_A - \Delta P)\pi r^2$
and on the circumferential area: $2\pi r L$

forces opposing pressure forces due to viscous effects: $\tau 2\pi r L$
where τ is the shear stress

According to Newton's second law of motion, the force in the x direction, $F_x = ma_x$. As noted earlier in this section, under fully developed flow conditions, there is no acceleration, or $a_x = 0$. Therefore, $F_x = 0$.

Thus, for the liquid element, all forces acting on it must balance, or,

$$P_A \pi r^2 - (P_A - \Delta P)\pi r^2 - \tau 2\pi r L = 0$$

or simplifying,

$$\frac{\Delta P}{L} = \frac{2\tau}{r} \qquad \text{Eq. (4)}$$

For Newtonian liquids, the shear stress is related to viscosity as seen earlier in Eq. (1). For pipe flow, we rewrite this equation in cylindrical coordinates as

$$\tau = \mu\left(-\frac{dV}{dr}\right)$$

Substituting Eq. (1) in Eq. (4)

$$\frac{dV}{dr} = -\left(\frac{\Delta P}{2\mu L}\right)r$$

$$\int dV = -\frac{\Delta P}{2\mu L}\int r dr$$

$$V(r) = -\left(\frac{\Delta P}{4\mu L}\right)r^2 + C_1$$

Where, C_1 is a constant.

For a viscous fluid flowing in a pipe, V=0 at r = R; therefore

$$C_1 = \frac{\Delta P}{4\mu L} R^2$$

Therefore, the velocity profile for a laminar, fully developed flow, in a horizontal pipe, is:

$$V(r) = \frac{\Delta P}{4\mu L}\left(R^2 - r^2\right)$$

or

$$V(r) = \frac{\Delta P R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad \text{Eq. (5)}$$

Therefore, for fully developed flow conditions we obtain a parabolic velocity profile. Furthermore, from Eq. (5), substituting $r = 0$, the maximum velocity, $V_{\max}$, is obtained at the pipe centerline, or

$$V_{\max} = \frac{\Delta P R^2}{4\mu L} \qquad \text{Eq. (6)}$$

Let us determine the volumetric flow rate by integrating the velocity profile across the cross-section of the pipe. First, we will examine a small ring of thickness ***dr*** with an area ***dA***, where $dA = 2\pi r dr$, as shown in Fig. A. The velocity, ***V***, in this thin annular ring is assumed to be constant. Then the volumetric flow rate through the annular ring, $\dot{V}_{ring}$ is

$$\dot{V}_{ring} = V(r)dA = V(r)2\pi r dr$$

$$\dot{V} = \int V(r)dA = \int_{r=0}^{r=R} V(r)2\pi r dr$$

$$\dot{V} = \frac{2\pi \Delta P R^2}{4\mu L}\int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r dr$$

$$\dot{V} = \frac{\pi R^4 \Delta P}{8\mu L} \qquad \text{Eq. (7)}$$

The mean velocity, $\bar{u}$, is defined as the volumetric flow rate divided by the cross-sectional area of the pipe, πR^2, or

$$\bar{u} = \frac{\dot{V}}{\pi R^2} \qquad \text{Eq. (8)}$$

or, substituting Eq. (7) in Eq. (8), we obtain

$$\bar{u} = \frac{\Delta P R^2}{8\mu L} \qquad \text{Eq. (9)}$$

Eq. (9) is called Poiseuille's Law.

If we divide Eq. (9) by (6), we obtain

$$\frac{\bar{u}}{V_{\max}} = 0.5 \ \text{(Laminar flow)} \qquad \text{Eq. (10)}$$

From Eq. (10), we observe that the average velocity is half the maximum velocity for fully developed laminar flow conditions. Furthermore, the radius (or diameter) of the pipe has a dramatic influence on the flow rate, as seen in Eq. (7). Doubling the diameter increases the volumetric flow rate 16-fold.

Now, divide Eq. (5) by Eq. (9)

i.e. $V(r) = \frac{\Delta P R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$ by $\bar{u} = \frac{\Delta P R^2}{8\mu L}$

$$\frac{V(r)}{\bar{u}} = 2\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad \text{Eq. (11)}$$

Eq. (11) represents the velocity profile of a Newtonian fluid flowing through a tube expressed in terms of the average velocity. The equation represents a parabola where the maximum velocity is $2\bar{u}$ at the center of the tube (r=0). Differentiating Eq. (11)

$$\frac{dV}{dr} = 2\bar{u}\left[\frac{2r}{R^2}\right]$$

The shear rate at the wall (r=R) for a Newtonian fluid is

$$-\left.\frac{dV}{dr}\right|_w = \frac{4\bar{u}}{R} \qquad \text{Eq. (12)}$$

(B) Velocity Profile and Shear Rate for a Power Law Fluid

From Eq. (4) and Eq. (2)

$$\frac{\Delta P}{L} = \frac{2\tau}{r} \ \text{and} \ \tau = K(\gamma)^n$$

$$\Rightarrow \tau = \frac{r\Delta P}{2L} = K(\gamma)^n$$

$$\frac{dV}{dr} = \left[\frac{\Delta P}{2LK}\right]^{1/n} (r)^{1/n}$$

Integrating and substituting the boundary condition, V = 0 at r = R:

$$V = \left[\frac{\Delta P}{2LK}\right]^{1/n} \left[\frac{1}{(1/n)+1}\right]\left[R^{(1/n)+1} - r^{(1/n)+1}\right] \qquad \text{Eq. (13)}$$

Eq. (13) represents the velocity profile of a power law fluid. The velocity profile equation will be more convenient to use if it is expressed in terms of the average velocity. Using a procedure similar to that used in the previous section the following expression for the average velocity can be derived

$$\bar{u}(\pi r^2) = \int_0^R 2\pi r \left[\frac{\Delta P}{2LK}\right]^{1/n} \left[\frac{n}{n+1}\right]\left[R^{(1/n)+1} - r^{(1/n)+1}\right] dr$$

Integrating and substituting limits:

$$\bar{u} = \left[\frac{\Delta P}{2LK}\right]^{1/n} [R]^{(n+1)/n} \left[\frac{n}{3n+1}\right]$$

The velocity profile in terms of the average velocity is

$$V = \bar{u}\left[\frac{3n+1}{n+1}\right]\left[1 - \left[\frac{r}{R}\right]^{(n+1)/n}\right]$$

Differentiating Equation (6.16) and substituting r = R for the shear rate at the wall:

$$-\left.\frac{dV}{dr}\right|_r = \bar{u}\left[\frac{3n+1}{n+1}\right]\left[\frac{n+1}{n}\right][\mathrm{R}]^{-(n+1)/n}[R]^{1/n}$$

$$-\left.\frac{dV}{dr}\right|_w = \bar{u}\left[\frac{3n+1}{n}\right]\left[\frac{1}{R}\right] \qquad \text{Eq. (14)}$$

Eq. (14) is a form of the Rabinowitsch-Mooney equation used to calculate shear rates for non-Newtonian fluids flowing through tubes. Converting into a form similar to Eq. (13):

$$-\left.\frac{dV}{dr}\right|_w = \frac{4\bar{u}}{R}\left[\frac{3}{4} + \frac{1}{4n}\right] \qquad \text{Eq. (15)}$$

Eq. (14) shows that the shear rate at the wall for a non-Newtonian fluid is similar to that for a Newtonian fluid except for the multiplying factor (0.75 + 0.25/n).

Problem 1

A glass capillary viscometer when used on a fluid with a viscosity of 10 centipoises allowed the fluid to efflux in 1.5 minutes. This same viscometer used on another fluid allowed an efflux time of 2.5 minutes. If the densities of the two fluids are the same, calculate the viscosity of the second fluid.

Solution:

Using Eq. (18) to solve for the viscometer constant:

$$k_v = \frac{\mu}{\rho t_e} = \frac{10}{1.5\rho}$$

For the second fluid:

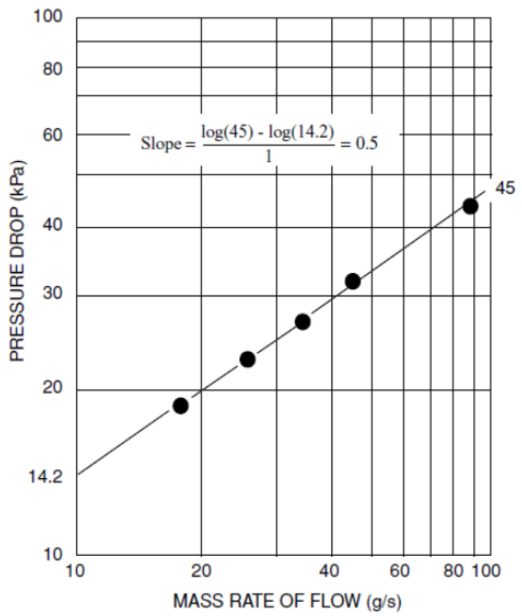
$$\mu = \rho \left[\frac{10}{1.5\rho} \right] (2.5) = 16.67 \text{ centipoise}$$

Problem 2

A tube viscometer having an inside diameter of 1.27 cm and a length of 1.219 m is used to determine the flow properties of a fluid having a density of 1.09 g/cm^3. The following data were collected for the pressure drop at various flow rates measured as the weight of fluid discharged from the tube. Calculate the flow behavior and consistency indices for the fluid.

Data with pressure drop (in KPa)	
(P_1-P_2)	Flow rate (g/s)
19.197	17.53
23.497	26.29
27.144	35.05
30.350	43.81
42.925	87.65

Solution



Log-log plot of pressure drop against mass rate of flow to obtain the flow behavior index from the slope.

Slope indicates n=0.5
Using Eq. (15) and Eq. (19)

Solving for τ_w

$$R = 0.5(1.27\text{ cm})(0.01\text{ m/cm}) = 0.00635\text{ m}$$

$$L = 1.219\text{ m}$$

$$\tau_w = \frac{[0.00635(0.5)]\Delta P}{1.219} = 0.002605\Delta P\text{ Pa}$$

The average velocity $\bar{u}$ in m/s can be calculated by dividing the mass rate of flow by the density and the cross-sectional area of the tube.

Let q = mass rate of flow in g/s.

$$\bar{u} = q\frac{\text{g}}{\text{s}}\frac{\text{cm}^3}{1.09\text{ g}}\frac{\text{m}^3}{(100)^3\text{ cm}^3}\frac{1}{\pi(0.00635)^2\text{ m}^2}$$

$$\gamma_w = \frac{4\bar{u}}{R}\left[\frac{3}{4}+\frac{1}{4n}\right] = \frac{4(0.007242\text{ q})}{0.00635}(1.25) = 5.7047\text{ q}$$

Problem 3

A single-cylinder rotational viscometer with a 1-cm radius and 6-cm length is being used to measure liquid viscosity. The following torque readings were obtained at several values of revolutions per minute (rpm):

N_r (rpm)	Ω (x 10^{-3} N Cm)
3	1.2
6	2.3
9	3.7
12	5.0

Compute the viscosity of the liquid based on the information provided.
Solution

Equation (26) requires the following input data (for example):

Ω = 2.3 x 10^{23} N cm = 2.3 x 10^{25} N m
N_r =6 rpm = 0.1 rev/s
L = 6 cm = 0.06 m
R_i = 1 cm = 0.01 m

Equation (26) is used to calculate viscosity from each rpm_torque reading combination.

$$\mu = \frac{(2.3 \times 10^{-5} Nm)}{8\pi^2 (0.1\,\text{rev/ sec})(0.06\,\text{m})(0.01\,\text{m})^2} = 0.485 \text{ Pa s}$$

Using the same approach, values of viscosity are obtained for each $N_r - \Omega$ combination

N_r (rev/s)	**Ω (x 10^{-5} N Cm)**	**μ (Pa s)**
0.05	1.2	0.507
0.1	2.3	0.485
0.15	3.7	0.521
0.2	5.0	0.528

Since it is assumed that the liquid is Newtonian, the four values can be used to compute an arithmetic mean of

μ = 0.510 Pa s