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Academic Scripts

Every linear programming problem (LPP) has another linear programming problem associated with it. The original problem is called the "***Primal***" while the other is called its "***Dual***".
Note: Either problem can be considered as the primal, while the another one will be treated as its dual.

The "*primal-dual*" share intimate relationship between them. The optimal solution of either problem reveals information about the optimal solution of the other.

Diet problem

The amounts of vitamins v_1 and v_2 per unit present in two different foods F_1 and F_2 respectively are given in Table 1.

Table 1

Vitamins	Food		Minimum daily requirements (units)
	F_1	F_2	
v_1	5	7	80
v_2	6	11	100
cost/unit	\$ 10	\$ 15	

The problem is to determine the minimum quantities of two foods F_1 and F_2 so that the minimum daily requirement of two vitamins is maintained at the minimum cost.

Mathematical formulation:

Let x_1 and x_2 be the number of units of food F_1 and F_2 to be procured respectively. We want to find the values of x_1 and x_2 so as to

$$\text{Mini } z_x = 10x_1 + 15x_2$$

subject to the constraints

$$5x_1 + 7x_2 \geq 80$$
$$6x_1 + 11x_2 \geq 100; \; x_1, x_2 \geq 0$$

In the constraints, "≥" sign indicates that taking more than minimum requirement is not harmful and purchase of negative quantity of food is meaningless. This LPP will be considered as the primal problem.
Now tuned with the above problem, let us consider a different problem.

Suppose a distributor sells two vitamins v_1 and v_2 with some other commodities. The retailer purchases the vitamins from distributor and two foods F_1 and F_2. The data is given in Table 2.

Table 2

Food	Vitamins		cost/unit
	v_1	v_2	
F_1	5	6	\$ 10
F_2	7	11	\$ 15
Minimum daily requirements (units)	80	100	

The problem of distributor is to maximize the profit by setting suitable selling price for the two vitamins v_1 and v_2 in such a way that the resulting prices of foods F_1 and F_2 do not exceed the existing market prices.
Mathematical formulation:

Let w_1 and w_2 be the number of units of vitamins v_1 and v_2 respectively. We want to find the values of w_1 and w_2 so as to

$$\text{Maximize } z_w = 80w_1 + 100w_2$$

subject to the constraints

$$5w_1 + 6w_2 \leq 10$$
$$7w_1 + 11w_2 \leq 15; \; w_1, w_2 \geq 0$$

This twin LPP is considered as the dual of the given primal.
From these two LPP's, we observe that

(1) The costs associated with the objective function of one problem are the requirement in the other's set of constraints.

(2) The constant coefficient matrix associated with one problem is the transpose of the constraint coefficient matrix associated with the other.

(3) The objective of one LPP is minimization while that of the other is maximization.

The relationship between primal-dual is illustrated in figure 1.

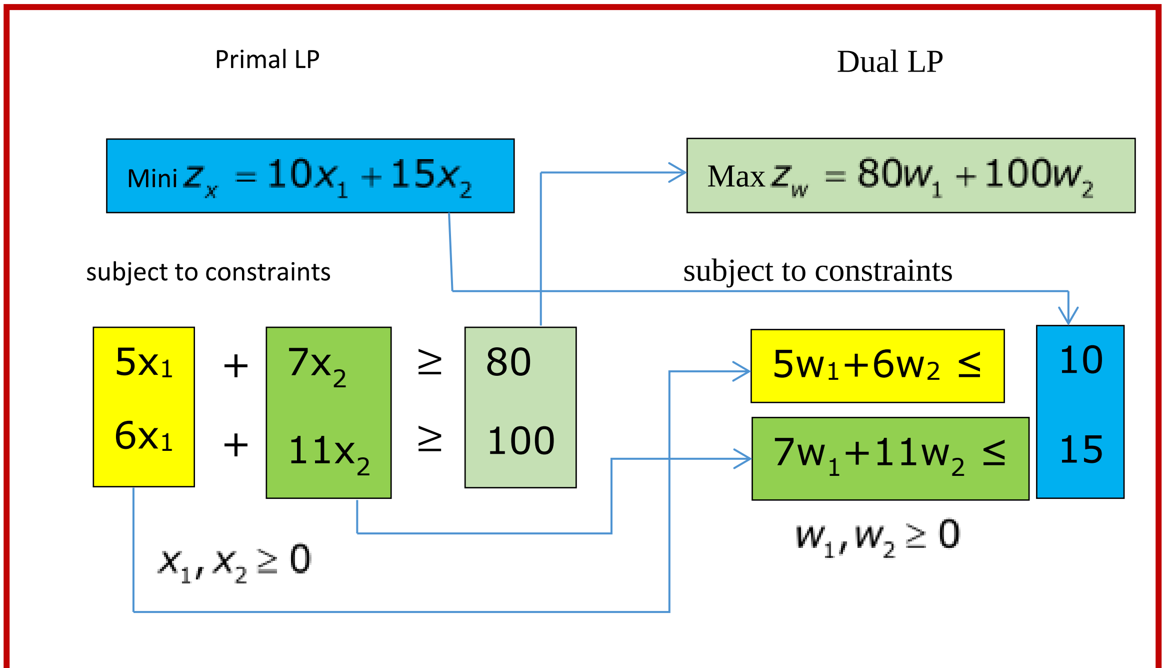

Fig. 1 Primal – Dual relationship

Now, we are ready to define primal-dual problems.

Matrix form of Symmetric primal and its dual

Primal problem: Determine a column vector $X \in \mathbb{R}^n$ to

maxi. $z_x = CX,\ C \in \mathbb{R}^n$ (primal objective function)

subject to $AX \le b, b \in \mathbb{R}^m, X \ge 0$ where A is an $m \times n$ real matrix.

Dual problem: Determine a column vector $W \in \mathbb{R}^m$ to

mini. $z_w = b^T W,\ b \in \mathbb{R}^m$ (dual objective function)

subject to $A^T W \le C^T, C \in \mathbb{R}^n, W \ge 0$ where A^T, b^T, C^T are the transposes of A, b, C respectively given in the primal.

Matrix form of Unsymmetric primal and its dual

Primal problem: Find a column vector $X \in \mathbb{R}^n$ to

maxi. $z_x = CX,\ C \in \mathbb{R}^n$ (primal objective function)

subject to $AX = b, b \in \mathbb{R}^m, X \ge 0$ where A is an $m \times n$ real matrix.

Dual problem: Determine a column vector $W \in \mathbb{R}^m$ to

mini. $z_w = b^T W,\ b \in \mathbb{R}^m$ (dual objective function)

subject to $A^T W = C^T, C \in \mathbb{R}^n, W$ is unrestricted in sign.

Algorithm for converting primal into its dual

If the constraints in a given LPP are equations and inequalities; non-negative variables or unrestricted variables, then the dual of the given LP can be obtained by performing following steps:

Step 1: First write the objective function in maximization form, if not.

Step 2: If a constraint has a sign "≥", then multiply both sides by – 1 and make the sign "≤".

Step 3: If a constraint has an equality sign, then replace it by two constraints involving inequalities as ≤ and ≥.

For example, a constraint equation $3x_1 + 2x_2 = 6$ should be replaced by two opposite inequalities as $\begin{cases} 3x_1 + 2x_2 \le 6 \\ 3x_1 + 2x_2 \ge 6 \end{cases}$.

The ≥ - type constraint should be converted to ≤ by multiplying both sides with -1 as $-3x_1 - 2x_2 \le -6$.

Step 4: Every unrestricted variable is replaced by the difference of two non-negative variables.

Note: The dual variables corresponding to primal equality constraints are unrestricted in sign and those corresponding with the primal inequalities are non-negative.

Step 5: We have the standard form of a given LPP in which

(i) all the constraints have "≤" sign when the objective function is of maximization form; or
(ii) all the constraints have "≥" sign when the objective function is of minimization form.

Step 6: Finally, the dual of the given problem is obtained by

(i) transposing the rows and columns of constraints coefficients;
(ii) transposing the cost coefficients of the objective function with the resources; the right side of the constraints;
(iii) changing the inequalities from "≤" to "≥" sign; and
(iv) minimising the objective function instead of maximizing it.

Duality Theorems: (without proof)

(a) The dual of the dual of a given primal is the primal.
(b) If X is any feasible solution to the primal problem and W is any feasible solution to the dual problem, then value of the objective function of the primal at X is less than or equal to that of the dual at W.

(c) If X^* is a feasible solution to the primal and W^* is a feasible solution to the dual such that $CX^* = b^T W^*$, the X^* is an optimal solution to the primal and W^* is an optimal solution to the dual.

One can combine (b) and (c) as

(d) If $X_0(W_0)$ is an optimal solution to the primal (dual), then there exist a feasible solution $W_0(X_0)$ to the dual (primal), such that $CX_0 = b^T W_0$. This is known as Basic Duality Theorem.

(e) Fundamental Duality Theorem

(i) If either the primal or the dual problem has a finite optimal solution, then the dual problem also has a finite optimal solution.
Furthermore, the optimal values of the objective functions of both the problems are same.

(ii) If either problem has an unbounded solution, then the other problem has infeasible solution.

(iii) Both problems may not have any solution.

(f) If the $k^{th} -$constraint of the primal is anequality, then the dual variable w_k is unrestricted in sign. Illustration

$$\text{Maximize } z_x = 5x_1 + 12x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 5$$
$$2x_1 - x_2 = 2;\ x_1, x_2 \ge 0$$

The standard form of the primal is

$$\text{Maximize } z_x = 5x_1 + 12x_2$$

subject to the constraints

$$x_1 + 2x_2 \le 5$$
$$2x_1 - x_2 \le 2$$
$$-2x_1 + x_2 \le -2;\ x_1, x_2 \ge 0$$

Then the dual is

$$\text{Minimize } z_w' = 5w_1 + 2w_2' - 2w_2''$$

subject to the constraints

$$w_1 + 2w_2' - 2w_2'' \ge 5$$
$$2w_1 - w_2' + w_2'' \ge 12;\ w_1, w_2', w_2'' \ge 0$$

Writing $w_2 = w_2^{'} - w_2^{"}$, the dual problem can be written as

$$\text{Minimize } z_w = 5w_1 + 2w_2$$

subject to the constraints

$$w_1 + 2w_2 \geq 5$$
$$2w_1 - w_2 \geq 12;\ w_1 \geq 0, w_2 \text{ unrestricted in sign}$$

(g) If the k^{th} – variable of the primal is unrestricted in sign, the k^{th} – constraint of dual is an equality. Illustration

$$\text{Maximize } z_x = 5x_1 + 12x_2$$

subject to the constraints

$$x_1 + 2x_2 \leq 5$$
$$2x_1 - x_2 \leq 2;\ x_1 \geq 0,\ x_2 \text{ unrestricted in sign}$$

Write unrestricted variable $x_2 = x_2^{'} - x_2^{"}, x_2^{'}, x_2^{"} \geq 0$

The standard form of the primal is

$$\text{Maximize } z_x = 5x_1 + 12x_2^{'} - 12x_2^{"}$$

subject to the constraints

$$x_1 + 2x_2^{'} - 2x_2^{"} \leq 5$$
$$2x_1 - x_2^{'} + x_2^{"} \leq 2;\ x_1, x_2^{'}, x_2^{"} \geq 0$$

Then the dual is

$$\text{Minimize } z_w = 5w_1 + 2w_2$$

Subject to the constraints

$$w_1 + 2w_2 \geq 5$$
$$2w_1 - w_2 \geq 12$$
$$-2w_1 + w_2 \geq -12;\ w_1, w_2 \geq 0$$

Combining second and third constraints, the dual problem can be written as

$$\text{Minimize } z_w = 5w_1 + 2w_2$$

Subject to the constraints

$$w_1 + 2w_2 \geq 5$$
$$2w_1 - w_2 = 12;\ w_1, w_2 \geq 0$$

(h) Complementary Slackness Theorem: For the optimal feasible solutions of the primal and dual problems

(1) if the inequality occurs in the i^{th} $-$relation of either system, the corresponding slack or surplus variable s_{n+i} is positive, then the i^{th} $-$variable w_i of its dual is zero.

(2) if the j^{th} $-$ variable is positive for either system, the j^{th} $-$ dual constraint holds as a strict equality. That is, the corresponding slack/surplus variable w_{m+j} is zero.

From (a) – (h), we summarize the correspondence between primal and its dual in Table 3.

Table 3

	Primal Problem	Dual Problem
1	Objective function is maximization	Objective function is minimization
2	Requirement vector	Price vector
3	Coefficient matrix A	Transpose of Coefficient matrix A^T
4	Constraints with ≤ sign	Constraints with ≥ sign
5	Relation	Variable
6	i^{th} $-$inequality	i^{th} $-$variable $w_i \geq 0$
7	i^{th} $-$constraint an equality	i^{th} $-$variable unrestricted in sign
8	Variable	Relation
9	If i^{th} $-$variable $x_i > 0$	i^{th} $-$relation strictly equality
10	If i^{th} $-$variable unrestricted in sign	i^{th} $-$constraint strict equality
11	If i^{th} $-$slack variable positive	i^{th} $-$variable is zero
12	If i^{th} $-$variable is zero	i^{th} $-$ surplus variable positive
13	Finite optimal solution	Finite optimal solution equal to that of the primal
14	Unbounded solution	Infeasible solution

Obtaining Solution of Dual from that of the Primal

Consider

Primal problem	Dual problem
Maximize $z_x = 40x_1 + 50x_2$ subject to the constraints	Minimize $z_W = 3w_1 + 5w_2$ subject to the constraints

$2x_1 + 3x_2 \le 3$ $8x_1 + 4x_2 \le 5$ $x_1, x_2 \ge 0$	$2w_1 + 8w_2 \ge 40$ $3w_1 + 4w_2 \ge 50$ $w_1, w_2 \ge 0$

The optimal solutions using simplex method for both the problems are

		C_j	40	50	0	0			C_j	-3	-5	0	0
B.V.	C_B	X_B	x_1	x_2	s_1	s_2	B.V.	C_B	W_B	w_1	w_2	S_1	S_2
x_2	50	7/8	0	1	1/2	1/8	w_2	-5	5/4	0	1	-3/16	1/8
x_1	40	3/16	1	0	-1/4	-3/4	w_1	-3	15	1	0	1/4	-1/2
$z_x = \frac{205}{4}$		$z_j - c_j$	0	0	15	5/4	$z'_w = -\frac{205}{4}$		$z_j - C_j$	0	0	3/16	7/8

In solution, B.V. stands for basic variable.

Advantages of Duality

The dual is important for the following reasons.

- The solution may be easier to obtain of the dual problem than the primal problem when the number of variables in the primal problem is less than the number of constraints.
- Duality frequently occurs in economics and other fields. In economics, it is used in the formulation of the input and output systems.
- The economical interpretations of the dual are useful in making future decisions in the financial activities.

Summary

- For every LPP, there is another LPP called the primal and dual respectively. When the primal is maximization problem, the dual is a minimization problem and vice versa.
- If the primal has an optimal solution, then the dual also has an optimal solution with same value of the objective functions.
- If the primal has an infeasible solution, the dual will have an unbounded solution and vice versa.
- The net evaluator values (ignoring minus signs) corresponding to the slack/surplus variables of the primal problem represent optimal values of the dual variables.
- The net evaluator values (ignoring minus signs) corresponding to the slack/surplus variables indicate the marginal profitability or shadow prices of the resources/variables they are related to. Thus, the shadow price of a given constraint of an LPP is the amount by which the optimal value of the objective function is improved if the right hand side of the constraint is increased by one unit.