[Academic Script]
[Solution of Difference Equations]

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Solution of Difference Equations

In this talk, let us discuss a few methods to solve difference equations and its application.

1. Solution of difference equation

The general form of difference equation is $y_{x+1} = Ay_x + B$ (1)

where A and B are constants and coefficient of y_{x+1} is one. Then the homogeneous solution can be computed by setting $B = 0$.

Then $y_{x+1} = Ay_x$

Substitute $y_x = \beta^x$. So, $\beta^{x+1} = A\beta^x \Rightarrow \beta = A$.

Thus, the homogeneous solution is $y_x = cA^x$, where c is a constant.The particular solution is

$$y_x = \begin{cases} B\left(\dfrac{1-A^x}{1-A}\right), A \neq 1 \\ Bx \qquad\qquad, A = 1 \end{cases} \tag{2}$$

Hence, the general solution is

$$y_x = \begin{cases} cA^x + B\left(\dfrac{1-A^x}{1-A}\right), A \neq 1 \\ c + Bx \qquad\qquad\quad, A = 1 \end{cases} \tag{3}$$

where $x = 0, 1, 2, \ldots$

Let us see the procedure to obtain this solution.

Let y_0 be given. Then

$$y_1 = Ay_0 + B$$
$$y_2 = Ay_1 + B = A^2 y_0 + AB + B$$
$$y_3 = Ay_2 + B = A^3 y_0 + A^2 B + AB + B$$

In general,

$$y_n = A^n y_0 + (A^{n-1} + A^{n-2} + \ldots + A + 1)B$$
$$= A^n y_0 + B\left(\frac{1-A^x}{1-A}\right),\ A \neq 1$$

Example: Solve $3y_{x+1} = 6y_x + 9, x = 0,1,2,\ldots$

Make coefficient of y_{x+1} to be 1. So $y_{x+1} = 2y_x + 3$.

Using (3), the solution is $y_x = c2^x + 3\left(\frac{1-2^x}{1-2}\right)$.

Let $y_0 = 7$. Then $7 = c2^0 + 3\left(\frac{1-2^0}{1-2}\right) \Rightarrow c = 7$.

Hence, the general solution is

$$y_x = 7\cdot 2^x + 3\left(\frac{1-2^x}{1-2}\right) = 10\cdot 2^x - 3$$

2. Linear Second-order difference equation with constant coefficients

Let the equation be

$$y_{x+2} + Ay_{x+1} + By_x = f_x$$

Here, f_x can be taken as a constant or a function of x. We discuss some cases here to find the solution of the difference equation.

Case 1: Let $f_x = A^x$, A is a constant. Then the particular solution is

$$y_x = cA^x \qquad (2)$$

Example: Solve $y_{x+2} - 5y_{x+1} + 6y_x = 4^x$.

The auxiliary equation $\beta^{x+2} - 5\beta^{x+1} + 6\beta^x = 0$ has roots $\beta_1 = 2$ and $\beta_2 = 3$. Then the solution is

$$y_x = c_1 2^x + c_2 3^x$$

For particular solution, let $y_x = c4^x$. Substitute in the given equation

$$c\cdot 4^{x+2} - 5c\cdot 4^{x+1} + 6c\cdot 4^x = 0 \Rightarrow 16c - 20c + 6c = 1 \Rightarrow c = \frac{1}{2}$$

Thus, the particular solution is $y_x = \frac{1}{2}4^x$. The general solution is

$$y_x = c_1 2^x + c_2 3^x + \frac{1}{2} 4^x$$

Example: Solve $y_{x+2} - 4y_{x+1} + 3y_x = 3^x$.

Check that the homogeneous solution is $y_x = c_1 + c_2 3^x$ which is the same as the function f_x i.e. 3^x. In such a case where the homogeneous solution includes a term similar to the function f_x, the particular solution is $y_x = cx3^x$

Then given equation gives

$$c(x+2)3^{x+2} - 4c(x+1)3^{x+1} + 3cx3^x = 3^x \Rightarrow c = \frac{1}{6}$$

So, the particular solution is $y_x = \frac{1}{6} x3^x$ and the general solution is

$$y_x = c_1 + c_2 3^x + \frac{1}{6} x3^x.$$

Example: Solve $y_{x+2} - 6y_{x+1} + 9y_x = 3^x$.

The auxiliary equation $\beta^{x+2} - 6\beta^{x+1} + 9\beta^x = 0$ has equal roots $\beta_1 = \beta_2 = 3$. So, the homogeneous solution is

$$y_x = c_1 3^x + c_2 x 3^x$$

We need to proceed as above example because homogeneous solution includes 3^x which is f_x. Let particular solution be $y_x = cx^2 3^x$. Substituting this in the given equation gives $c = \frac{1}{18}$. Hence, the particular solution is $y_x = \frac{1}{18} x^2 3^x$ and the general solution is

$$y_x = c_1 3^x + c_2 x 3^x + \frac{1}{18} x^2 3^x$$

Case 2: For $f_x = x^n$.

Let a particular solution be $y_x = A_0 + A_1 x + ... A_n x^n$.

The method is similar as given in case 1.

To find homogeneous solution (say)

$$y_x = c_1 \beta_1^x + c_2 \beta_2^x$$

if equation is of a second order.

For particular solution use the procedure outlined in case 1 and illustrated in different examples.

Example: Solve $y_{x+2} - 4y_{x+1} + 3y_x = x^2$.

The roots of auxiliary equation are $\beta_1 = 1$ and $\beta_2 = 3$. So, the homogeneous solution is

$$y_x = c_1 + c_2 3^x$$

Since, the degree of the difference equation is 2, we assume particular solution as $y_x = A_0 + A_1 x + A_2 x^2$.

This has a constant A_0. The homogeneous solution also has a constant c_1, so we multiply particular solution by x and get

$$y_x = A_0 x + A_1 x^2 + A_2 x^3$$

Then given equation becomes

$$-6A_2 x^2 - 4A_1 x + (-2A_0 + 4A_2) = x^2$$

Equating coefficients gives

$$-6A_2 = 1$$
$$4A_1 = 0$$
$$-2A_0 + 4A_2 = 0$$

Therefore, $A_0 = -\frac{1}{2}, A_1 = 0, A_2 = -\frac{1}{6}$.

Then the particular solution is $y_x = -\frac{1}{2}x - \frac{1}{6}x^2$ and the general solution is $y_x = c_1 + c_2 3^x - \frac{1}{2}x - \frac{1}{6}x^2$.

Case 3: For $f_x = A^x + Bx^n$

Both cases discussed above will be used. Let us illustrate with following example.

Example: Solve $y_{x+2} - 4y_{x+1} + 3y_x = 5^x + 2x$.

The roots of auxiliary equation are $\beta_1 = 1$ and $\beta_2 = 3$. So, the homogeneous solution is

$$y_x = c_1 + c_2 3^x$$

The particular solution for $f_x = 5^x$ is

$$y_x = c5^x$$

The particular solution for $f_x = 2x$ is

$$y_x = A_0 + A_1 x$$

Arguing as above, we take

$$y_x = A_0 x + A_1 x^2$$

Thus, the combined particular solution is

$$y_x = c5^x + A_0 x + A_1 x^2$$

Substitute in given equation. We get

$$-2A_0 - 4A_1 x + 8c5^x = 5^x + 2x$$

Equating coefficients, we get $A_0 = 0, A_1 = -\frac{1}{2}, c = \frac{1}{8}$. Thus, the particular solution is

$$y_x = -\frac{1}{2}x^2 + \frac{1}{8}5^x$$

and the general solution is

$$y_x = c_1 + c_2 3^x - \frac{1}{2}x^2 + \frac{1}{8}5^x$$

Case 4: $f_x = A^x x^n$

The particular solution takes the form $y_x = A^x(A_0 + A_1 x + \ldots + A_n x^n)$. Follow the steps discussed above to solve difference equation.

3. Economic Illustration

Let G_t be government expenditure

C_t be consumption expenditure

I_t be induced private investment.

Then, national income is

$$Y_t = G_t + C_t + I_t \quad (1)$$

Define relations between them as

$$C_t = aY_{t-1} \quad (2)$$

$$I_t = b(C_t - C_{t-1}) = ab(Y_{t-1} - Y_{t-2}) \quad (3)$$

$$G_t = 1 \quad (4)$$

a in eq.(2) is the marginal tendency to consume and b in eq.(3) is called the relation. With these eq.(1) becomes

$$Y_t - a(1+b)Y_{t-1} + abY_{t-2} = 1 \quad (5)$$

We have second-order linear difference equation. The objective is to get Y_t in terms of a and b. For homogeneous solution, substitute $Y_t = \beta^t$. The auxiliary equation is $\beta^2 - a(1+b)\beta + ab = 0$. Let two roots be β_1 and β_2. with $\beta_1 > \beta_2$. Then

$$\begin{aligned} \beta_1 &= \frac{1}{2}\left[a(1+b) + \sqrt{a^2(1+b)^2 - 4ab}\right] \\ \beta_2 &= \frac{1}{2}\left[a(1+b) - \sqrt{a^2(1+b)^2 - 4ab}\right] \end{aligned} \quad (6)$$

So, the homogeneous solution is $Y_t = c_1\beta_1^t + c_2\beta_2^t$ where c_1 and c_2 are constants to be determined using initial conditions.

For particular solution, we assume $Y_t = c_3 1^t = c_3$. Then eq. (5) becomes

$$c_3 - a(1+b)c_3 + abc_3 = 1 \Rightarrow c_3 = \frac{1}{1-a}, 0 < a < 1$$

So, the particular solution is $Y_t = \dfrac{1}{1-a}$ and the general solution is

$$Y_t = c_1\beta_1^t + c_2\beta_2^t + \frac{1}{1-a}$$

Take initial conditions: At $t = 0$, $Y_t = 0$ and $t = 1$, $Y_t = 1$.

$$c_1 + c_2 + \frac{1}{1-a} = 0$$

$$c_1\beta_1 + c_2\beta_2 + \frac{1}{1-a} = 0$$

Hence, $c_1 = \frac{\beta_2 - a}{(1-a)(\beta_1 - \beta_2)}$ and $c_2 = \frac{a - \beta_1}{(1-a)(\beta_1 - \beta_2)}$.

Let us illustrate its application with a = 0.5 and b = 0. So β_1 = 0.5 and β_2 = 0 which gives c_1 = - 2 and c_2 = 0. The functional form of government income is $Y_t = 2 + (-2)(0.5)^t$.

Note: Nature of the roots β_1 and β_2 determines the government's national income.

Summary

- The solution of second order difference equation is sum of homogeneous function and particular solution.
- The homogeneous function can be obtained by setting RHS of difference equation to be zero.
- The particular solution depends on functional form of g_x.
- General solution of the difference equation is sum of homogeneous function and particular solution.