[Academic Script]

[Elementary Difference Equations & Their Applications to Economics]

Subject:	Business Economics
Course:	B.A., 6th Semester, Undergraduate
Paper No. & Title:	Paper – 631 Advanced Mathematical Techniques
Unit No. & Title:	Unit - 4 Difference Equations
Lecture No. & Title:	1: Elementary Difference Equations & Their Applications to Economics

Elementary Difference Equations

and

Their Applications to Economics

It is not always possible to find exact solution of differential equation. So, we need to find another tool to obtain approximate numerical solution. The finite difference method is one such tool. When the variable time is to be treated as discrete, we use difference equation.

In this talk, let us discuss a few elementary concepts concerning finite difference.

Finite Differences:

Consider, a function $y = f(x)$. The derivatives of $f(x)$ is defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Let Δx be a fixed quantity. Then, write above equation as

$$f(x+\Delta x) - f(x) = y(x+\Delta x) - y(x) = \Delta y(x)$$

Δ is the operator acting on y and it is known as a difference operator. The fixed quantity Δx is called the difference interval.

Thus, we have a relationship

$$\Delta y(x) = y(x+\Delta x) - y(x) \qquad \text{...(1)}$$

which means that we take a difference interval Δx from the point x and find the difference between two values of y at the points x and $x+\Delta x$.

This denotes that the distance between any two consecutive points in the domain are a finite distance apart. Consider this finite distance to be a constant h.

Thus, if we have one point x and $\Delta x = h$ then the points $x, x+h, x+2h, ...$ forms an arithmetic progression.

If $h=1$, the successive points starting from x will be

$$x, x+1, x+2, \ldots$$

Δy of equation (1) is $\Delta^2 y(x) = \Delta y(x+h) - \Delta y(x)$ called the second difference. Repeating the process, we have

$$\Delta(\Delta y(x)) = \Delta y(x+h) - \Delta y(x)$$

$$\text{or} \quad \Delta^2 y(x) = \Delta y(x+h) - \Delta y(x)$$

which is known as the second difference. This is usually written as

$$\Delta^2 y(x) = \Delta y(x+h) - \Delta y(x)$$

Now, $\Delta y(x+h) = y(x+2h) - y(x+h)$

So, $$\begin{aligned}\Delta^2 y(x) &= y(x+2h) - y(x+h) - y(x+h) + y(x) \\ &= y(x+2h) - 2y(x+h) + y(x)\end{aligned}$$

Similarly,

$$\Delta(\Delta^2 y(x)) = y(x+3h) - 3y(x+2h) + 3y(x+h) - y(x)$$

In general,

$$\Delta^n y(x) = (-1)^0 \begin{pmatrix} n \\ 0 \end{pmatrix} y(x+nh) + (-1)^1 \begin{pmatrix} n \\ 1 \end{pmatrix} y(x+(n-1)h) + \ldots + (-1)^{n-1} \begin{pmatrix} n \\ n-1 \end{pmatrix} y(x+h) + (-1)^n y(x) \ldots (2)$$

where, $\begin{pmatrix} n \\ m \end{pmatrix} = \dfrac{n!}{m!(n-m)!} \qquad \begin{pmatrix} n \\ 0 \end{pmatrix} = 1 \qquad 0! = 1$

Example 1. Find the first difference of $y(x) = x^2$ when $x = 2$.

Solution:

$$\begin{aligned}\Delta y(x) &= y(x+1) - y(x) \\ &= y(3) - y(2) \\ &= 3^2 - 2^2 \\ &= 5\end{aligned}$$

.

Example 2. Find the first difference of function $y(x) = x(x-1)$ at $x = 3$

Solution:

$$
\begin{aligned}
y(x) &= x^2 - x \\
\Delta y(x) &= y(x+1) - y(x) \\
&= y(4) - y(3) \\
&= 4^2 - 4 - 3^2 + 3 \\
&= 16 - 4 - 9 + 3 \\
&= 16 - 4 - 9 + 3 \\
&= 6
\end{aligned}
$$

From equation (2), we can write

$$
\begin{aligned}
y(x+1) &= y(x) + \Delta y(x) \\
y(x+2) &= y(x+1) + \Delta y(x+1) \\
&= y(x) + \Delta y(x) + (\Delta y(x) + \Delta^2 y(x)) \\
&= y(x) + 2\Delta y(x) + \Delta^2 y(x) \\
y(x+3) &= y(x) + 3\Delta y(x) + 3\Delta^2 y(x) + \Delta^3 y(x)
\end{aligned}
$$

In general,

$$y(x+n) = y(x) + \binom{n}{1}\Delta y(x) + \binom{n}{2}\Delta^2 y(x) + \ldots + \binom{n}{n-1}\Delta^{n-1} y(x) + \binom{n}{n}\Delta^n y(x) \qquad \ldots(3)$$

2. Some Operators

i) The operator Δ is defined as

$\Delta y(x) = y(x+1) - y(x)$

$\Delta^2 y(x) = \Delta y(x+1) - \Delta y(x)$ and so on.

ii) The operator E

The operator E is defined as

$$Ey(x) = y(x+h)$$

For $h=1$,

$$\begin{aligned} Ey(x) &= y(x+1) \\ E^2 y(x) &= E[E(y(x))] \\ &= E(y(x+1)) \\ &= y(x+2) \end{aligned}$$

In general,

$$E^n y(x) = y(x+n), \qquad n = 1, 2, 3, \dots$$

For $n = 0,\ E^0 y(x) = y(x).$

iii) Relation between Δ and E

$$\begin{aligned} \Delta y(x) &= y(x+1) - y(x) \\ &= Ey(x) - y(x) \\ &= (E-1)y(x) \\ \therefore \Delta &= E - 1 \end{aligned}$$

For any n,

$$\begin{aligned} \Delta y(x+n) &= y(x+n+1) - y(x+n) \\ &= Ey(x+n) - y(x+n) \\ &= (E-1)y(x+n) \\ \therefore \Delta &= E - 1 \end{aligned}$$

iv) The relation $E = e^D$

Taylor series expansion is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$$

$$\Rightarrow y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \dots$$

Denote $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2$,

Then for $h=1$, we get

$$\begin{aligned} y(x+1) &= y(x) + Dy(x) + \frac{D^2}{2!}y(x) + \dots \\ &= (1 + D + \frac{D^2}{2!} + \dots)y(x) \\ &= e^D y(x) \end{aligned}$$

But,

$$\begin{aligned} &y(x+1) = Ey(x) \\ &\therefore Ey(x) = e^D y(x) \\ &\Rightarrow E = e^D = (1+\Delta) \end{aligned}$$

3. Difference Equations

Difference equations are similar to differential equations.

An equation which shows the relation between the independent variable x, the dependent variable y and its finite difference is known as difference equation.

Order of difference Equation

The maximum difference of the difference intervals of the equation defines order of a difference equation.

$y(x+3) + y(x+2) - y(x) = x$ (order 3)

Because,

$$\begin{aligned} &y(x+3) = y(x) + 3\mathrm{D}y(x) + 3\mathrm{D}^2 y(x) + \mathrm{D}^3 y(x) \\ &y(x+2) = y(x) + 2\mathrm{D}y(x) + \mathrm{D}^2 y(x) \end{aligned}$$

Substituting in above equation, we get

$$\begin{aligned} &2y(x) + 5\Delta y(x) + 4\Delta^2 y(x) + \Delta^3 y(x) - y(x) = x \\ &\therefore y(x) + 5\Delta y(x) + 4\Delta^2 y(x) + \Delta^3 y(x) = x \end{aligned}$$

This, establishes that the order of the equation is 3 which is the highest difference.

New notation:

For simplicity, we will write

$$y(x+3) = y_{x+3}$$

Then, the above equation can be written as

$$y_{x+3} + y_{x+2} - y_x = x$$

4. Solution

Consider the difference equation

$$y_{x+n} - y_x = n, \qquad x = 0,1,2,\ldots(1)$$

A function y of x is a solution of a difference equation, if every value of y satisfies the difference equation for all values of the independent variable x.

Let $y_x = x \qquad \ldots(2)$

Then for $x+n$,

$y_{x+n} = x+n$

Substitute in the difference equation (1)

$$LHS = (x+n) - x = n = RHS$$

The function (2) satisfies the difference equation (1) for $x = 0,1,2,\ldots$

So, the function (2) is a solution of (1).

Consider the function $y_x = x + k$, k is a constant.

Substituting in (1) gives

$$LHS = (x + k + n) - (x + k) = n = RHS$$

$\therefore (3)$ is a solution of (1). In fact, (3) is the general solution and (2) is the particular solution when $k = 0$. Graphically, (3) gives us a family of straight lines parallel to $y = x$. This means given a point on the graph, all the other y- values will be uniquely determined for every x that is considered. Equivalently, the function y is uniquely determined.

Consider a second order linear difference equation as

$$y_{x+2} - 4y_{x+1} + 4y_x = 2x.$$

When two y- values are given for two consecutive x- values, there is one and only one solution y for every value of x.

Theorem: For a linear difference equation of order n, when n y- values are given for n consecutive x- values, there is one and only one solution y for every value of x that is being considered.

Note: The theorem is stated for linear difference equation.

Note: When a solution of the difference equation that satisfies the given initial condition is obtained, it is unique.

5. Homogenous linear difference equation with constants coefficients

The general form of a homogenous linear difference equation with constant coefficients of order n is

$$y_{x+n} + A_1 y_{x+n-1} + \ldots + A_{n-1} y_{x+1} + A_n y_x = 0 \qquad \ldots (1)$$

Set $y_x = \beta^x$ where β is constant.

Then equation (1) becomes $(\beta^n + A_1\beta^{n-1} + \ldots + A_n)\beta^x = 0$. $\beta^n + A_1\beta^{n-1} + \ldots + A_n = 0$ is called the auxiliary or characteristic equation.

The roots of this equation will be solution of (1).

The general solution is

$$y_x = c_1\beta_1^{\ x} + c_2\beta_2^{\ x} + \ldots + c_n\beta_n^{\ x} \quad \ldots(2)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Case 1. Linear homogenous difference equation with constant coefficients of the first order.

$$y_{x+1} - A_1 y_x = 0$$

Let $y_x = \beta^x$ .Then

$$\begin{aligned} &\beta^{x+1} - A_1\beta^x = 0 \\ &\Rightarrow \beta^x(\beta - A_1) = 0 \\ &\Rightarrow \beta = A_1 \end{aligned}$$

Thus, the solution of the difference equation is $y_x = c_1 A_1^{\ x}$.

Example: Solve $y_{x+1} - 2y_x = 0$.

Let $y_x = \beta^x$ .Then the characteristic equation is

$$\begin{aligned} &\beta^{x+1} - 2\beta^x = 0 \\ &\Rightarrow \beta^x(\beta - 2) = 0 \\ &\Rightarrow \beta = 2 \end{aligned}$$

Thus, the solution is $y_x = c_1 2^x$.

Case 2. Linear homogenous difference equation with constant coefficient of order 2.

The form of the equation is $y_{x+2} + A_1 y_{x+1} + A_2 y_x = 0$

With substitution $y_x = \beta^x$, the auxiliary equation is $\beta^2 + A_1\beta + A_2 = 0$.

We need to discuss three different cases depending on the roots.

5.1 When the two roots β_1 and β_2 are real and distinct.

The solution is $y_x = c_1\beta_1^{\ x} + c_2\beta_2^{\ x}$.

Example: Solve and check $y_{x+2} - 7y_{x+1} + 12y_x = 0$.

Solution: The auxiliary equation is

$$\beta^2 - 7\beta + 12 = 0$$

$$\beta_1 = 3, \beta_2 = 4$$

Therefore, the general solution is $y_x = c_1 3^x + c_2 4^x$.

Check:

$$y_{x+2} = c_1 2^{x+2} + c_2 4^{x+2}$$

$$= 4c_1 2^x + 16c_2 4^x$$

$$6y_{x+1} = 6c_1 2^{x+1} + 6c_2 4^{x+1}$$

$$= 12c_1 2^x + 24c_2 4^x$$

$$8y_x = 8c_1 2^x + 8c_2 4^x$$

LHS of given equation

$$= 4c_1 2^x + 16c_2 4^x - 12c_1 2^x - 24c_2 4^x + 8c_1 2^x + 8c_2 4^x = 0 = \text{RHS}$$

5.2 When two roots are equal.

The solution is

$$y_x = c_1\beta^x + c_2 x\beta^x$$

If the order of the difference equation is 3 and all the roots of the auxiliary equation are equal, then the general solution is

$$y_x = c_1\beta^x + c_2 x\beta^x + c_3 x^2\beta^x$$

Note: Solutions for higher order equations with equal roots can be obtained in a similar way.

Example: Solve $y_{x+2} - 6y_{x+1} + 9y_x = 0$

Solution: The auxiliary equation with substitution $y_x = \beta^x$ is

$$\beta^2 - 6\beta + 9 = 0$$

Then roots are $\beta_1 = \beta_2 = 3$.

The general solution of the difference equation is $y_x = c_1 3^x + c_2 x 3^x$.

Note: When the auxiliary equation of order three has two of the roots to be equal then the solution is

$$y_x = c_1\beta_1^x + c_2\beta_2^x + c_3 x\beta_3^x$$

Example: Solve $y_{x+3} - 5y_{x+2} + 8y_{x+1} - 4y_x = 0$.

Solution: The auxiliary equation is

$$\begin{aligned}
&\beta^3 - 5\beta^2 + 8\beta - 4 = 0 \\
&\Rightarrow (\beta-1)(\beta-2)^2 = 0 \\
&\Rightarrow \beta_1 = 1, \beta_2 = \beta_3 = 2
\end{aligned}$$

Thus, the solution is $y_x = c_1 + c_2 2^x + c_3 x 2^x$.

Case 3. When the roots are conjugate complex numbers:

Let the roots be

$$\beta_1 = a + ib = r(\cos\theta + i\sin\theta)$$

$$\beta_2 = a - ib = r(\cos\theta - i\sin\theta)$$

where $r = \sqrt{a^2 + b^2}$ $, \theta = \tan^{-1}\dfrac{b}{a}$

The solution is

$$y_x = d_1\beta_1^x + d_2\beta_2^x$$

where d_1 and d_2 are complex conjugates.

Let $d_1 = m + in, d_2 = m - in$

$$d_1\beta_1^x = d_1 r^x(\cos\theta + i\sin\theta)^x = d_1 r^x(\cos\theta x + i\sin\theta x)$$

$$d_2\beta_2^x = d_2 r^x(\cos\theta x - i\sin\theta x)$$

Thus,

$$y_x = r^x[(d_1 + d_2)\cos\theta x + i(d_1 - d_2)\sin\theta x] = r^x[c_1\cos\theta x + c_2\sin\theta x]$$

where $\begin{aligned} c_1 &= d_1 + d_2 = 2m \\ c_2 &= i(d_1 - d_2) = -2n \end{aligned}$

Thus, c_1 and c_2 are real numbers and we have y_x as a real number.

Example: Solve $y_{x+2} + 2y_{x+1} + 4y_x = 0$

Solution: The auxiliary equation is

$$\beta^2 + 2\beta + 4 = 0$$

The roots are $\beta_1 = -1 + \sqrt{3}i, \beta_2 = -1 - \sqrt{3}i$

i.e. $a = -1, \; b = \sqrt{3}$

Thus, $r = 2 \quad \theta = \dfrac{2\pi}{3}$

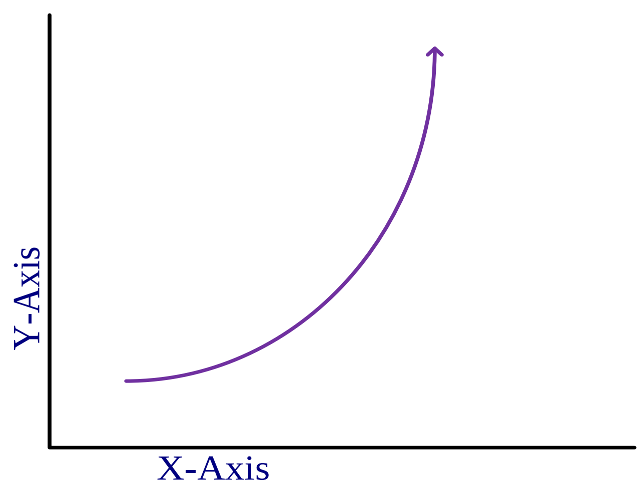
So, the general solution is $y_x = 2^x\left(c_1 \cos\dfrac{2\pi x}{3} + c_2 \sin\dfrac{2\pi x}{3}\right)$.

6. Geometrical Interpretation of Solution

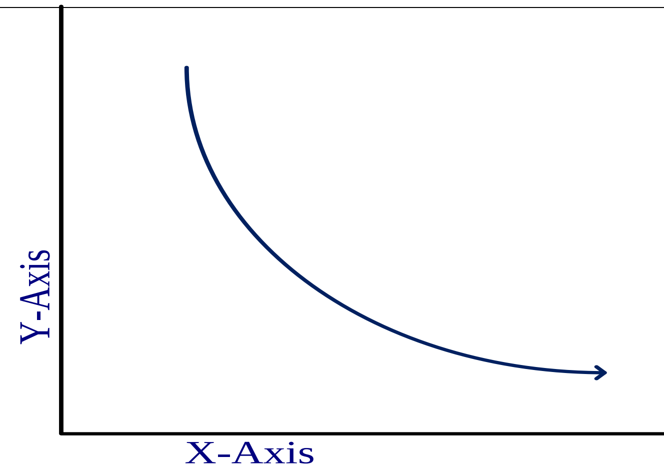
We have seen that when $\beta_1 \neq \beta_2$, the solution is $y_x = c_1\beta_1^x + c_2\beta_2^x$

As c_1 and c_2 are constants, the nature of y_x when $x \to \infty$ depends on the values of β_1 and β_2. The larger one will determine the behaviour of y_x. Let β_1 be the larger root.

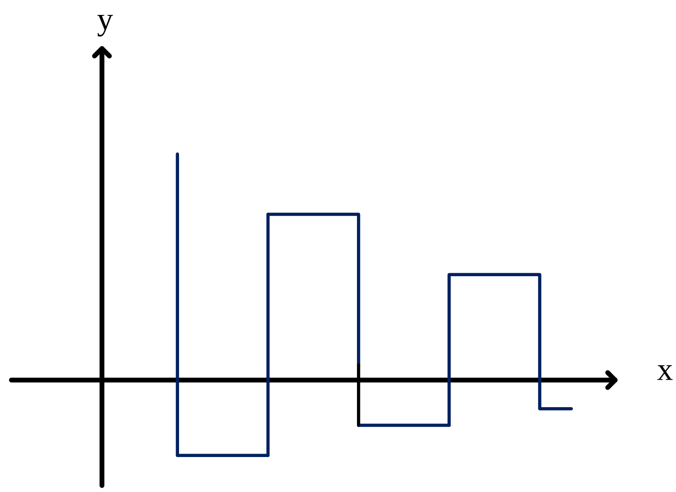
i) When $c_1 > 0, \beta_1 > 1$ the graph is



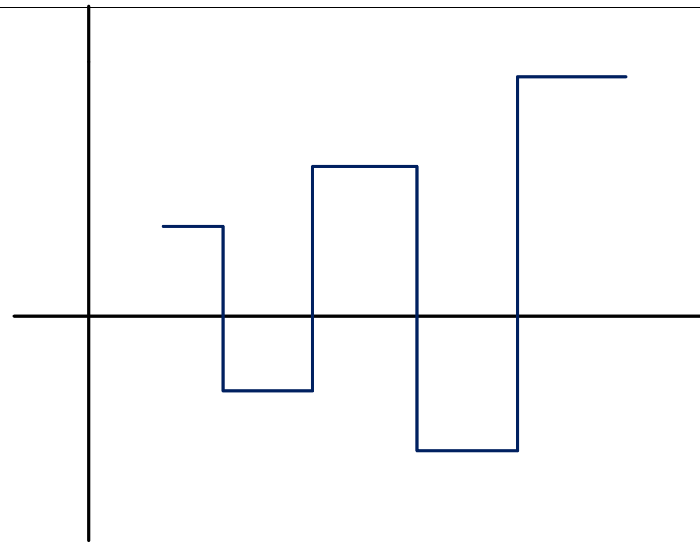
ii) When $c_1 > 0, 0 < \beta_1 < 1$ the curve is



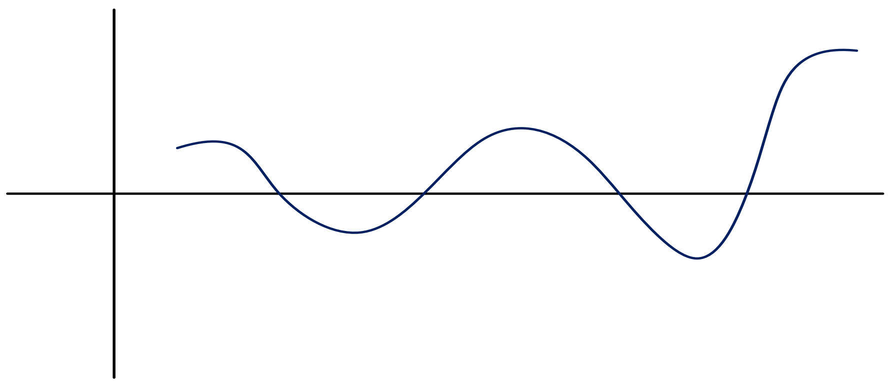
iii) When $c_1 > 0,\ -1 < \beta < 0$ the curve is



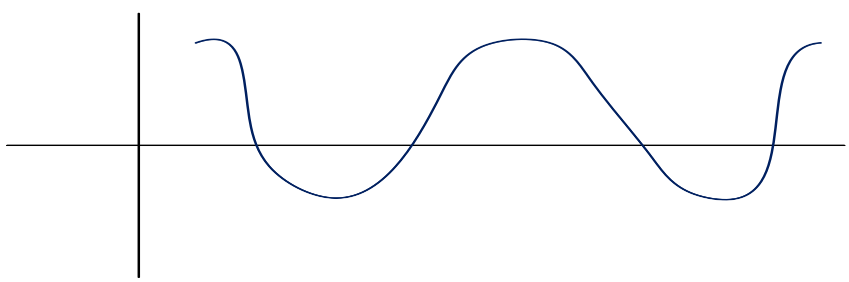
iv) When $c_1 > 0,\ \beta < -1$ the curve is



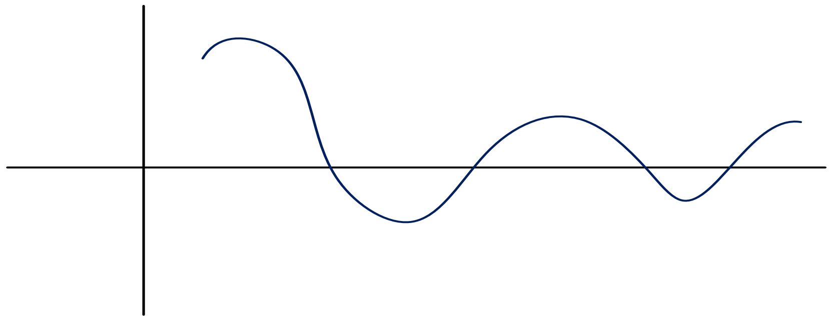
v) When roots are complex and $r > 1$, we get explosive oscillations as



vi) For $r = 1$, the curve is simple harmonic as



vii) For $r < 1$, the curve is damped oscillations as



Summary

- A *difference equation* is an equation that contains a dependent variable, independent variable and its finite difference.
- The maximum difference of the difference intervals of the equation is the *order* of a difference equation.
- A function y *of* x is called a *solution* of a difference equation if every value of y satisfies the difference equation for all values of the independent variable x.
- General solution of the difference equation is sum of homogeneous solution and particular solution.