[Academic Script]

[Concept of Convexity/Concavity]

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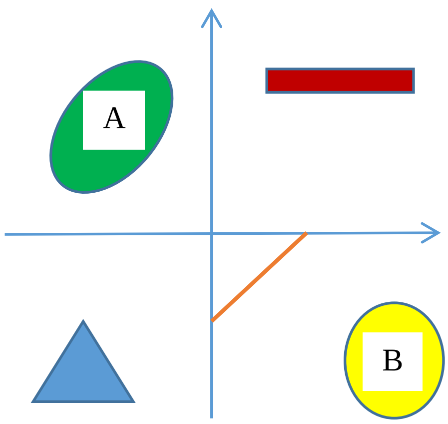
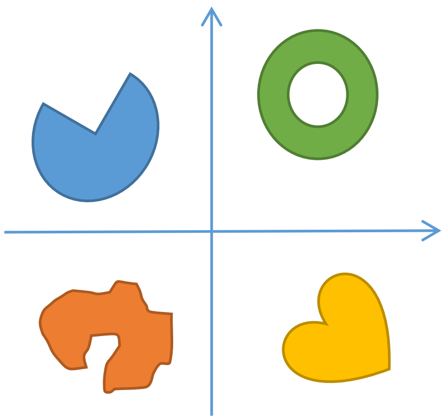
Academic Script

Concept of Convexity/Concavity

Convex Set: A set $X \subset \mathbb{R}^n$ is convex if for every pair of points x_1 and $x_2 \in X$ and any $\lambda \in [0, 1]$, the point $x = \lambda x_1 + (1- \lambda)\, x^2$ also belongs to the set X. Equivalently, a set is convex if every point on the line segment between every pair of points in the set is also in the set.



Convex sets in $\mathbb{R}^2$



Non-convex sets in $\mathbb{R}^2$
Non-Convex sets in $\mathbb{R}^2$

Strictly - convex Set: A set $X \subset \mathbb{R}^n$ is strictly convex, if for every pair of points x_1 and $x_2 \in X$ and any $\lambda \in (0, 1)$, x is an interior point of X where $x = \lambda x_1 + (1- \lambda)x_2$.

In figure, the sets A and B are strictly convex.

Note: We exclude the cases $\lambda = 0$ and $\lambda = 1$ because x_1 or x_2 could be a boundary point of a set.

Convex function: The function f is convex if

$f(x) \leq \lambda f(x_1) + (1- \lambda)f(x_2)$

where $x = \lambda x_1 + (1- \lambda)x_2$ and $\lambda \in [0, 1]$.

Note: It is strictly convex if the strict inequality holds when $\lambda \in (0, 1)$.

Concave function: The function f is concave if

$f(x) \geq \lambda f(x_1) + (1- \lambda)f(x_2)$

where $x = \lambda x_1 + (1- \lambda)x_2$ and $\lambda \in [0, 1]$.

Note: It is strictly concave if the strict inequality holds when $\lambda \in (0, 1)$.

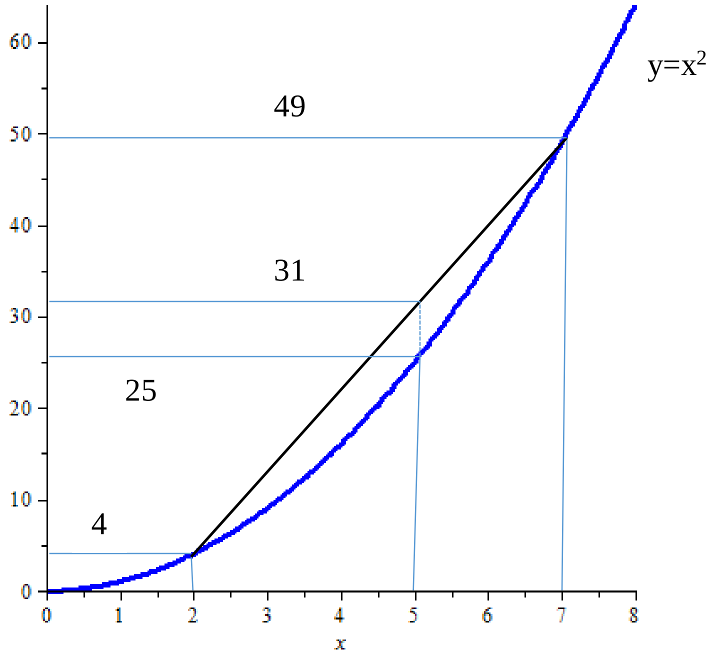
Let us illustrate this concept for the function $f(x) = x^2$.

Let $\lambda = 0.4$, $x_1 = 2$ and $x_2 = 7$.

We have $x = \lambda x_1 + (1- \lambda)x_2 = (0.4)2 + (0.6)7 = 5$

So, $f(x) = f(5) = 25$

which is the height of the function at the point x in fig.



Now, from convex combination, we get a straight line connecting f(2) and f(7) as

$0.4\ (2)^2 + 0.6\ (7)^2 = 31$

which is the height of a straight line connecting the points (2, 4) and (7, 49) at x = 5. Hence, $f(x) = x^2$ is strictly convex between these two points.

Note: One can prove that in general, $f(x) = x^2$ is strictly convex.

Level Set: A level set of the function $y = f(x_1, x_2, \ldots, x_n)$ is the set

$L = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : f(x_1, x_2, \ldots, x_n) = c\}$

for some real number c.

The level set shows the set of points in the domain of the function that gives equal values of the function.

In economics, level sets are used in Consumer theory (where they are called indifference curves), Producer theory (where they are called isoquants) etc.

Quasiconcavity is concerned with the shape of the level sets of the function.

Better Set: The better set of a point $(x_{10}, x_{20}, \ldots, x_{n0})$ is

$B(x_{10}, x_{20}, \ldots, x_{n0}) = \{(x_{10}, x_{20}, \ldots, x_{n0}) : f(x_1, x_2, \ldots, x_n) \geq f(x_{10}, x_{20}, \ldots, x_{n0})\}$.

Note: The better set of a point is the set of the points in the domain that yields at least as large a function value.

Quasiconcave function: A function f with domain $X \subset \mathbb{R}^n$ is quasiconcave, if every point in X, the better set B of that point is a convex set. It is strictly quasiconcave if B is strictly convex.

The shapes of the level sets of quasiconcave function will depend on the direction in which the function increases.

If the function is increasing, then the level sets must have negative slope and the convexity of the better sets will depict the shape.

Note: If the function is increasing in one variable and decreasing in the other or decreasing in both the variables, the quasiconcavity will generate different shapes for the level sets.

Worse set: The worse set of a point $(x_{10}, x_{20}, \ldots, x_{n0})$ is

$W(x_{10}, x_{20}, \ldots, x_{n0}) = \{(x_{10}, x_{20}, \ldots, x_{n0}) : f(x_1, x_2, \ldots, x_n) \leq f(x_{10}, x_{20}, \ldots, x_{n0})\}$.

Quasiconvex function: A function $f(x_1, x_2, \ldots, x_n)$ with domain $X \subset \mathbb{R}^n$ is quasiconvex if every $(x_{10}, x_{20}, \ldots, x_{n0}) \in X$, the worse $W(x_{10}, x_{20}, \ldots, x_{n0})$ is a convex set. It is strictly Quasiconvex if $W(x_{10}, x_{20}, \ldots, x_{n0})$ is strictly convex.

Note: Any convex function is quasiconvex but not vice versa.

Curvature of a function is described by the second-order derivatives.

We have studied that for $f'' > 0$ the function is convex, which means that for $f' > 0$, the function increases more rapidly as x increases while for $f' < 0$, the function value falls less quickly.

For $f'' < 0$ the function is concave, which means that for $f' > 0$, the function increases less quickly as x increases while for $f' < 0$, the function value falls more quickly.

The second-order differential for the function z = f(x,y) is

$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$

This suggests that d^2z depends on the cross-partial second derivative f_{xy} as well as on f_{xx} and f_{yy}.

We have following sufficient conditions for a function to be strictly convex or strictly concave.

Theorem: If the function $z = f(x,y)$ defined on $\mathbb{R}^2$ is twice continuously differentiable and $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 > 0$ whenever at least one of the dx or dy is non-zero, then $z = f(x,y)$ is a strictly convex function.

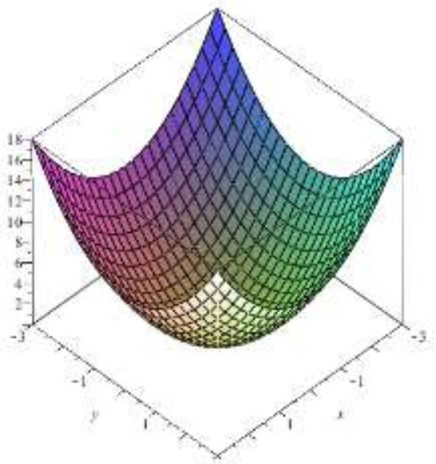
Theorem: If the function $z = f(x,y)$ defined on $\mathbb{R}^2$ is twice continuously differentiable and $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 < 0$ whenever at least one of the dx or dy is non-zero, then $z = f(x,y)$ is a strictly concave function.

Example: Show that the function $z = x^2 + y^2$ is strictly convex.

Clearly, $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 2$. Then

$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = 2dx^2 + 2dy^2$

Since, $dx^2 \geq 0$, $dy^2 \geq 0$ and both are zero only if $dx = dy = 0$, then the function is strictly convex.

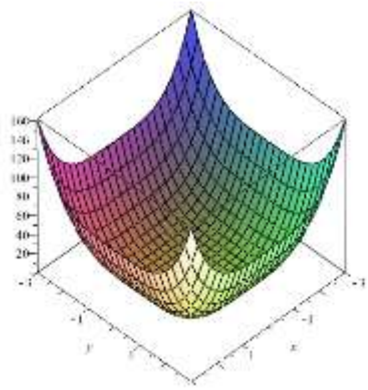


Example: For $z = x^4 + y^4$

$f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = 0$

and so $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = 12x^2dx^2 + 12y^2dy^2$

which takes the value zero when $x = y = 0$, then the function is strictly convex.



Theorem: If the function z = f(x,y) defined on $\mathbb{R}^2$ is twice continuously differentiable then it is convex if and only if

$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \geq 0$.

Theorem: If the function z = f(x,y) defined on $\mathbb{R}^2$ is twice continuously differentiable then it is concave if and only if

$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \leq 0$.

Note: For weak convexity or weak concavity, the condition holds with weak inequality.

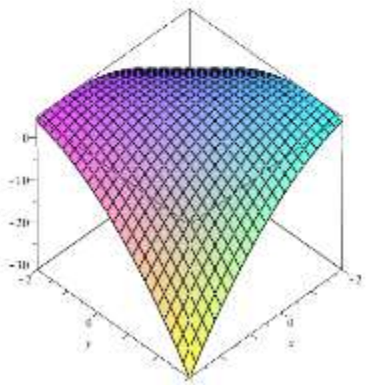
Example: The function $z = 5 – (x + y)^2$ is concave.

We have $f_x = - 2 (x + y)$, $f_y = - 2 (x + y)$

$f_{xx} = - 2$, $f_{yy} = - 2$, $f_{xy} = - 2$.

Then $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = - 2 (dx + dy)^2 \leq 0$.

So the function is concave as shown in figure.



Theorem: For any function y = f(x), $x \in \mathbb{R}^n$ which is twice continuously differentiable with Hessian matrix H, it follows that

(1) the function f is strictly convex on $\mathbb{R}^n$ if H is positive definite for all $x \in \mathbb{R}^n$, i.e. $d^2y = dx^THx > 0$.

(2) the function f is strictly concave on $\mathbb{R}^n$ if H is negative definite for all $x \in \mathbb{R}^n$, i.e. $d^2y = dx^THx < 0$.

(3) the function f is convex on $\mathbb{R}^n$ if H is positive semi-definite for all $x \in \mathbb{R}^n$, i.e. $d^2y = dx^T Hx \geq 0$.

(4) the function f is concave on $\mathbb{R}^n$ if H is negative semi-definite for all $x \in \mathbb{R}^n$, i.e. $d^2y = dx^T Hx \leq 0$.

Note: The conditions on H are only sufficient in the case of strict convexity / concavity, while the conditions are both necessary and sufficient in case of (weak) convexity / concavity.

Theorem: Let H be the Hessian matrix associated with a twice continuously differentiable function $y = f(x)$, $x \in \mathbb{R}^n$. Then

(1) H is a positive definite on $\mathbb{R}^n$ if and only if its leading principal minors are positive; $|H_1| > 0$, $|H_2| > 0, \ldots, |H_n| = |H| > 0$ for $x \in \mathbb{R}^n$. In this case $d^2y > 0$ and so f is strictly convex.

(2) H is a negative definite on $\mathbb{R}^n$ if and only if its leading principal minors alternate in signs beginning with a negative value. i.e. $|H_1| < 0$, $|H_2| > 0, \ldots, |H_n| = |H| > 0$, if n is even and < 0 if n is odd for $x \in \mathbb{R}^n$. In this case $d^2y < 0$ and so f is strictly concave.

(3) H is a positive semi-definite on $\mathbb{R}^n$ if and only if its leading principal minors are positive or zero; $|H_1| \geq 0$, $|H_2| \geq 0, \ldots, |H_n| = |H| \geq 0$ for $x \in \mathbb{R}^n$. In this case $d^2y \geq 0$ and so f is convex.

(4) H is a negative semi-definite on $\mathbb{R}^n$ if and only if its leading principal minors alternate in signs beginning with a negative or zero. In this case $d^2y \leq 0$ and so f is concave.

Example: Consider $z = (x + y)^{1/2}$.

The second-order partial derivatives are

$f_{xx} = -\frac{1}{4}(x+y)^{-3/2}$, $f_{xy} = f_{yx} = -\frac{1}{4}(x+y)^{-3/2}$, $f_{yy} = -\frac{1}{4}(x+y)^{-3/2}$

Then $|H_1| = f_{xx} < 0$, $|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 0$.

Since, $|H_2| = 0$, we check for weak concavity. $|H_1| = f_{xx}$, f_{yy} are both negative and $|H_2| \geq 0$. Therefore, f is concave.

Bordered Hessian matrix: Let f be defined on $\mathbb{R}^n$ possess continuous first – and second – order partial derivatives. The bordered Hessian of the function f is

$$H_b = \begin{bmatrix} 0 & f_{x1} & \cdots & f_{xn} \\ f_{x1} & f_{x1x1} & \cdots & f_{x1xn} \\ \vdots & \vdots & \vdots & \vdots \\ f_{xn} & f_{xnn1} & \cdots & f_{xnxn} \end{bmatrix}$$

Note that the bordered Hessian matrix is formed by taking the Hessian matrix and adding [0 f_{x1} f_{x2} … f_{xn}] as a first column and a first row. Then

$$|H_{b1}| = \begin{vmatrix} 0 & f_{x1} \\ f_{x1} & f_{x1x1} \end{vmatrix}$$

$$|H_{b2}| = \begin{vmatrix} 0 & f_{x1} & f_{x2} \\ f_{x1} & f_{x1x1} & f_{x1x2} \\ f_{x2} & f_{x2x1} & f_{x2x2} \end{vmatrix}$$, and so on.

Note: $|H_{b1}| = -f_1^2$ and so must be non-positive.

Theorem: Suppose that f is a function defined on $\mathbb{R}^n$ possess continuous first – and second – order partial derivatives. Let H_b represents the bordered Hessian of f. Then

(1) If $|H_{b2}| > 0$, $|H_{b3}| < 0$, … $|H_{bn}| > 0$ (if n even) and < 0 (if n odd) for all $x \in \mathbb{R}^n$, then f is quasiconcave.

(2) If $|H_{b2}| < 0$, $|H_{b3}| < 0$, … $|H_{bn}| < 0$ for all $x \in \mathbb{R}^n$, then f is quasiconvex.

Note: Any convex function is quasiconvex but vice versa may not hold.

Example: Show that the function $f(x,y) = xy^2$ defined on $\mathbb{R}^2$ is quasiconcave.

$$|H_{b2}| = \begin{vmatrix} 0 & f_x & f_y \\ f_x & f_{xx} & f_{xy} \\ f_y & f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & y^2 & 2xy \\ y^2 & 0 & 2y \\ 2xy & 2y & 2x \end{vmatrix} = 6xy^4 > 0 \text{ for } x, y > 0$$

Therefore f is quasiconcave.

Summary

- The shapes of the level sets of quasiconcave function will depend on the direction in which the function increases.
- If the function is increasing in one variable and decreasing in the other or decreasing in both the variables, the quasiconcavity will generate different shapes for the level sets.

- Any convex function is quasiconvex but not vice versa.
- Curvature of a function is described by the second-order derivatives.
- For $f'' > 0$ the function is convex, which means that for $f' > 0$, the function increases more rapidly as x increases while for $f' < 0$, the function value falls less quickly.
- For $f'' < 0$ the function is concave, which means that for $f' > 0$, the function increases less quickly as x increases while for $f' < 0$, the function value falls more quickly.
- If the function $z = f(x,y)$ defined on $\mathbb{R}^2$ is twice continuously differentiable and $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 > 0$ whenever at least one of the dx or dy is non-zero, then $z = f(x,y)$ is a strictly convex function.
- If the function $z = f(x,y)$ defined on $\mathbb{R}^2$ is twice continuously differentiable and $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 < 0$ whenever at least one of the dx or dy is non-zero, then $z = f(x,y)$ is a strictly concave function.
- The conditions on Hessian are only sufficient in the case of strict convexity / concavity, while the conditions are both necessary and sufficient in case of (weak) convexity / concavity.
- The bordered Hessian matrix is formed by taking the Hessian matrix and adding $[0 \quad f_{x1} \quad f_{x2} \ldots f_{xn}]$ as a first column and a first row.