[Academic Script]

[Partial Differentiation of Functions of Function and Implicit Function]

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Academic Script

We have studied total differentiation of u = f(x,y) where x and y were independent variables and also independent of each other.

In this talk, we first consider the case of dependent variable.

Let x = f(t), y = h(t), where t is the independent variable. Thus, u is a function of x and y; x and y are functions of t.

We try to answer "What is the derivative of u with respect to t?"

Let u be wheat produced by x labor on y land. Now assume that the number of labor x and area of land y, depend on the amount of money t (say) that is on-hand. Thus, when there is a small increase in t, it will increase in x and y which results increase in u. So, we need to study how much change in u will occur with a unit change in u. Denote this by $\frac{du}{dt}$.

We know that $\frac{\partial u}{\partial x}$ is the change in u due to a small change in x keeping y constant. Also, $\frac{dx}{dt}$ denotes the change in x due to a small unit change in t. Thus, $\frac{\partial u}{\partial x}\frac{dx}{dt}$ will be the amount of change in u due to a small change in t that is transmitted through x. Similarly, $\frac{\partial u}{\partial y}\frac{dy}{dt}$ will be the amount of change in u due to a small change in t that is transmitted through y. Therefore, the change in u due to a small change in t will be linear sum of these two effects as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt}$$

This $\frac{du}{dt}$ is called the total derivative of u with respect to t.

Example: Find the total derivative of u with respect to t if

$u = x^2 + y^2$, $x = t^2$ and $y = t^2 + 1$.

We know that

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt}$$

where $u_x = 2x$, $u_y = 2y$, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2t$. So,

$\frac{du}{dt} = (2x)(2t) + (2y)(2t) = 4t\,(x + y)$

Example: Find $\frac{du}{dx}$ when $u = x^2 + y^2$ and $y = 2x$.

Set $y = 2t$ and $x = t$. So

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = 2x(1) + 2y(2) = 2x + 4y$$

Note: Since $x = t$, we have

$$\frac{du}{dx} = u_x \frac{dx}{dx} + u_y \frac{dy}{dx} = u_x + u_y \frac{dy}{dx}$$

(which can be used when x and y are not independent.)

Interpretation: The change in u is due to increment of x is the sum of u_x, which is the change in u when there is an increment in x treating y constant and change in u due to increment in y, which in turn is due to the change in x.

Example: Consider $V = \pi r^2 h$, where r is the radius, h is the height and V is volume of a cylinder. Let $r = t^2$ and $h = t^2$ where t is time. What will be the rate of increase in the volume of the cylinder per unit of time?

We want to compute

$$\frac{dV}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt}$$

Now, $V_r = 2\pi rh$ and $V_h = \pi r^2$. $\frac{dr}{dt} = 2t$ and $\frac{dh}{dt} = 2t$. Then

$\frac{dV}{dt} = (2\pi rh)(2t) + (\pi r^2)(2t) = 6\pi t^5$

Note: For $u = f(x, y, z, \ldots)$, $x = g(t)$, $y = h(t)$, …

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} + u_z \frac{dz}{dt} + \cdots$$

Two or More Independent Variables case:

Let $u = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$. Then

$$\frac{\partial u}{\partial t} = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial s} = u_x \frac{\partial x}{\partial s} + u_y \frac{\partial y}{\partial s}$$

where $\frac{\partial u}{\partial t}$ is the total derivative of u with respect to t when s is considered to be constant.

Example: Find $\frac{\partial u}{\partial t}$ if $u = x^2 + y^2$, $x = t^2 + s^2$, $y = t^2 - s^2$.

We have

$$\frac{\partial x}{\partial t} = 2t, \qquad \frac{\partial y}{\partial t} = 2t$$

$u_x = 2x$, $u_y = 2y$

So, $\frac{\partial u}{\partial t} = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t}$ = (2x)(2t) + (2y)(2t) = 4(x+y)t

Note: Result holds for n – variables.

Theorem: Given u = f(x,y). The differential of u is $du = f_x dx + f_y dy$ regardless of whether x and y are independent or not.

Proof: We have seen the result when x and y are independent. We need to show that the differential du holds when x and y are not independent.

Let u = f(x,y), x = g(s,t), y = h(s,t). Then

$$\frac{\partial u}{\partial s} = u_x \frac{\partial x}{\partial s} + u_y \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t}$$

Multiply first equation by ds and second by dt and add them. Then

LHS = $\frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt$.

Since, u = f(x,y) = f(g(s,t),h(s,t)). We set

$du = u_s ds + u_t dt$

RHS after addition is

$u_x\left[\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt\right] + u_y\left[\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt\right]$

But dx = $\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt$ and dy = $\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt$.

Therefore, RHS = $u_x dx + u_y dy$.

Combining LHS and RHS gives du = = $u_x dx + u_y dy$ which is differential of u = f(x,y).

Example: Let u = f(q_1, q_2) and $p_1q_1 + p_2q_2 = M$ where u is a utility function with goods q_1 and q_2. M denotes income and p_1 and p_2 are the prices of q_1 and q_2. The second equation is known as budget equation. Find the marginal utility of q_1 which is $\frac{\partial u}{\partial q_1}$ and also $\frac{\partial^2 u}{\partial q_1^2}$ which denotes the rate of change of marginal utility of q_1.

Write $q_2 = \frac{M-p_1q_1}{p_2}$. Then q_1 is the independent variable and we have

$$\frac{du}{dq_1} = u_{q_1} + u_{q_2}\frac{dq_2}{dq_1}$$

Now, $\frac{dq_2}{dq_1} = -\frac{p_1}{p_2}$. Therefore, $\frac{du}{dq_1} = u_{q_1} - u_{q_2}\frac{p_1}{p_2}$.

Student is asked to compute $\frac{\partial^2 u}{\partial q_1^2}$.

Higher-order Differentials

Let u = f(x,y), x = g(t), y = h(t) where t is the independent variable. Then

$$\frac{du}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt}$$

$$\frac{d^2u}{dt^2} = \frac{\partial}{\partial x}\left[u_x\frac{dx}{dt} + u_y\frac{dy}{dt}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[u_x\frac{dx}{dt} + u_y\frac{dy}{dt}\right]\frac{dy}{dt}$$

First term = $\frac{\partial}{\partial x}\left[u_x\frac{dx}{dt} + u_y\frac{dy}{dt}\right]\frac{dx}{dt}$

$$= \left[u_{xx}\frac{dx}{dt} + u_x\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) + u_{yx}\frac{dy}{dt} + u_y\frac{\partial}{\partial x}\left(\frac{dy}{dt}\right)\right]\frac{dx}{dt}$$

Here, $u_x\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) = u_x\frac{\partial}{\partial t}\left(\frac{dx}{dt}\right)\frac{dt}{dx} = u_x\frac{d^2x}{dt^2}\frac{dt}{dx}$

And $u_y\frac{\partial}{\partial x}\left(\frac{dy}{dt}\right) = 0$. Therefore,

First term = $u_{xx}\left(\frac{dx}{dt}\right)^2 + u_x\frac{d^2x}{dt^2}\frac{dt}{dx}\frac{dx}{dt} + u_{yx}\frac{dy}{dt}\frac{dx}{dt}$

$$= u_{xx}\left(\frac{dx}{dt}\right)^2 + u_x\frac{d^2x}{dt^2} + u_{yx}\frac{dy}{dt}\frac{dx}{dt}$$

Similarly, Second term $= u_{yy}\left(\frac{dy}{dt}\right)^2 + u_y\frac{d^2y}{dt^2} + u_{xy}\frac{dx}{dt}\frac{dy}{dt}$

So,

$$\frac{d^2u}{dt^2} = u_{xx}\left(\frac{dx}{dt}\right)^2 + u_{yy}\left(\frac{dy}{dt}\right)^2 + u_x\frac{d^2x}{dt^2} + u_y\frac{d^2y}{dt^2} + 2u_{xy}\frac{dx}{dt}\frac{dy}{dt}$$

Example: Given $u = x^2 + xy + y^2$, $x = t^3 + 1$, $y = t - t^3$.

$$\frac{du}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt}$$

$u_x = 2x + y$, $u_y = x + 2y$

$u_{xx} = 2$, $u_{xy} = 1$, $u_{yy} = 2$

$\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 1 - 3t^2$

$\frac{d^2x}{dt^2} = 6t$, $\frac{d^2y}{dt^2} = -6t$

$\frac{du}{dt} = (2x + y)\,3t^2 + (x + 2y)(1 - 3t^2)$

$$\frac{d^2u}{dt^2} = u_{xx}\left(\frac{dx}{dt}\right)^2 + u_{yy}\left(\frac{dy}{dt}\right)^2 + u_x\frac{d^2x}{dt^2} + u_y\frac{d^2y}{dt^2} + 2u_{xy}\frac{dx}{dt}\frac{dy}{dt}$$

$$= 30t^4 - 42t^2 + 6t + 2$$

Note: When $x = g(t)$ and $y = h(t)$ are linear $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = 0$.

Hence,

$$\frac{d^2u}{dt^2} = u_{xx}\left(\frac{dx}{dt}\right)^2 + u_{yy}\left(\frac{dy}{dt}\right)^2 + 2u_{xy}\frac{dx}{dt}\frac{dy}{dt}$$

Higher-order Derivatives When

$u = f(x, y)$, $y = g(x)$

where x is the independent variable and y is dependent variable. This type of functional form occurs most in economics.

We have

$$\frac{du}{dx} = u_x + u_y\frac{dy}{dx}$$

$$\frac{d^2u}{dx^2} = \frac{\partial}{\partial x}\left(u_x + u_y\frac{dy}{dx}\right) + \frac{\partial}{\partial y}\left(u_x + u_y\frac{dy}{dx}\right)\frac{dy}{dx}$$

$$= u_{xx} + u_{xy}\frac{dy}{dx} + u_y\frac{\partial}{\partial x}\left(\frac{dy}{dx}\right) + \left(u_{xy} + u_{yy}\frac{dy}{dx} + u_y\frac{\partial}{\partial y}\left(\frac{dy}{dx}\right)\right)\frac{dy}{dx}$$

$$= u_{xx} + 2u_{xy}\frac{dy}{dx} + u_y\frac{d^2y}{dx^2} + u_{xy} + u_{yy}\left(\frac{dy}{dx}\right)^2$$

Example: Find $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ for $u = x^2 + 2y^2$, $y = 4x - 1$.

We have $\frac{du}{dx} = u_x + u_y \frac{dy}{dx}$

$$\frac{d^2u}{dx^2} = u_{xx} + 2u_{xy}\frac{dy}{dx} + u_y\frac{d^2y}{dx^2} + u_{xy} + u_{yy}\left(\frac{dy}{dx}\right)^2$$

$\frac{dy}{dx} = 4$ and $\frac{d^2y}{dx^2} = 0$

$u_x = 2x$, $u_y = 4y$

$u_{xx} = 2$, $u_{xy} = 0$, $u_{yy} = 4$

Therefore, $\frac{du}{dx} = 2x + 16y$

$$\frac{d^2u}{dx^2} = 2 + 4(4)^2 + 2(0)(4) + 0 = 66$$

Extension to Three Variables

Given u = f(x,y,z) and z = g(x,y).

$$\frac{du}{dx} = u_x\frac{\partial x}{\partial x} + u_y\frac{\partial y}{\partial x} + u_z\frac{\partial z}{\partial x} = u_x + u_z\frac{\partial z}{\partial x}$$

$$\frac{d^2u}{dx^2} = \frac{\partial}{\partial x}\left(u_x + u_z\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial x} + \frac{\partial}{\partial y}\left(u_x + u_z\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial x} + \frac{\partial}{\partial z}\left(u_x + u_z\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial x}$$

But $\frac{\partial y}{\partial x}$ =0 and $\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)$ = 0. So

$$\frac{d^2u}{dx^2} = u_{xx} + u_{zz}\left(\frac{\partial z}{\partial x}\right)^2 + 2u_{xz}\frac{\partial z}{\partial x} + u_z\frac{\partial^2 z}{\partial x^2}$$

Similarly,

$$\frac{du}{dy} = u_y + u_z\frac{\partial z}{\partial y}$$

$$\frac{d^2u}{dy^2} = u_{yy} + u_{zz}\left(\frac{\partial z}{\partial y}\right)^2 + 2u_{yz}\frac{\partial z}{\partial y} + u_z\frac{\partial^2 z}{\partial y^2}$$

Example: Find $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ for $u = x^2 + y^2 + z^2$, $z = 3x - 2y$.

$$\frac{du}{dx} = u_x + u_z\frac{\partial z}{\partial x} = 2x + 2z(3) = 2x + 6z$$

$$\frac{d^2u}{dx^2} = u_{xx} + u_{zz}\left(\frac{\partial z}{\partial x}\right)^2 + 2u_{xz}\frac{\partial z}{\partial x} + u_z\frac{\partial^2 z}{\partial x^2}$$

$$= 2 + 2(3)^2 + 2(0)(3) + 2z\,(0) = 20$$

Homogeneous Function

Let us understand concept of homogeneous function through example.

Let $f(x,y) = x^2 - y^2$.

If we replace x by tx and y by ty in the function, where t is a positive constant, then

$f(tx,ty) = (tx)^2 - (ty)^2 = t^2(x^2 - y^2) = t^2f(x,y)$

For $f(x,y) = x^3 - y^3$, we have $f(tx,ty) = t^3f(x,y)$.

By induction, For $f(x,y) = x^n - y^n$, $f(tx,ty) = t^nf(x,y)$. Then we say that f is homogeneous.

For example if $f(x,y) = x^2 + xy - 3y^2$, then $f(tx,ty) = t^2f(x,y)$.

A function is said to be homogeneous function of degree n if $f(tx,ty) = t^nf(x,y)$.

Example: $f(x,y,z) = \frac{x+y}{z}$.

We have $f(tx,ty,tz) = \frac{tx+ty}{tz} = \frac{x+y}{z} = t^0f(x,y,z)$. So f is homogeneous function of degree 0.

Properties of Homogeneous Function

Consider a homogeneous function of degree n as

$u = f(x,y)$

Then $\qquad f(tx,ty) = t^nf(x,y)$

Since t takes any value, let t = 1/x. Then

$f(tx,ty) = f\left(1,\frac{y}{x}\right) = g\left(\frac{y}{x}\right)$

i.e. f(tx,ty) becomes a function g which has variable y/x. Also,

$t^nf(x,y) = (1/x)^nf(x,y)$

Comparing, last two form gives $f(x,y) = (1/x)^n g\left(\frac{y}{x}\right)$.

In general, if $u = f(x,y,z,\ldots)$

$f(tx,ty,tz,\ldots) = t^nf(x,y,z,\ldots)$

$f(x,y,z,\ldots) = x^ng\left(\frac{y}{x},\frac{z}{x},\ldots\right)$

$\qquad = y^nh\left(\frac{x}{y},\frac{z}{y},\ldots\right)$

$= z^n k\left(\frac{x}{z}, \frac{y}{z}, \ldots\right)$

Example: Consider a homogeneous function of degree 3 as f(x,y) $= x^2y - xy^2$.

Then $f(x,y) = x^3 g\left(\frac{y}{x}\right)$ where $g\left(\frac{y}{x}\right) = \frac{y}{x} - \frac{y^2}{x^2}$.

If f(x,y) is a homogeneous function of degree n, the k-th derivative of f is a homogeneous function of n – k provided the derivatives exists.

Consider $f(x,y) = x^2y^2 + xy^3$.

Then f is a homogeneous function of degree 4.

$f_x = 2xy^2 + y^3$ is a homogeneous function of degree 4 – 1 = 3.

$f_{xx} = 2y^2$ is a homogeneous function of degree 4 – 2 = 2.

Euler's Theorem

Euler's theorem is obtained from homogeneous function and useful for marginal productive theory. It is also known as adding-up theorem.

Let u = u(x,y) be a homogeneous function of degree n. Then

$xu_x + yu_y = nu$

Note: u_x and u_y are partial derivatives of u w.r.t x and y respectively.

Proof: u(x,y) is a homogeneous function of degree n, so

$u(tx,ty) = t^n u(x,y)$.

$\frac{\partial}{\partial t}(LHS) = \frac{\partial u(tx,ty)}{\partial tx}\frac{\partial tx}{\partial t} + \frac{\partial u(tx,ty)}{\partial ty}\frac{\partial ty}{\partial t} = xu_{tx} + yu_{ty}$

$\frac{\partial}{\partial t}(RHS) = \frac{\partial t^n}{\partial t}u(x,y) + t^n\frac{\partial u(x,y)}{\partial t} = nt^{n-1}u(x,y) + 0 = nt^{n-1}u(x,y)$

Therefore,

$xu_{tx} + yu_{ty} = nt^{n-1}u(x,y)$

As t is any number, put t = 1. So,

$xu_x + yu_y = nu(x,y)$

In general, for more than two variables,

$xu_x + yu_y + zu_z + \ldots = nu(x,y,z,\ldots)$

where n is the degree of homogeneous function.

Example: $u(x,y,z) = 3x + 2y - 4z$ is a linear homogeneous function.

Then by Euler's theorem

$xu_x + yu_y + zu_z = 1(3x + 2y - 4z)$

Check: $u_x = 3$, $u_y = 2$ and $u_z = -4$.

Therefore, LHS = $3x + 2y - 4z$ = RHS

Example: $u(x,y) = x^2 - 3y^2$ is a homogeneous function of degree 2. By Euler's theorem,

$xu_x + yu_y = 2(x^2 - 3y^2)$

Check: $u_x = 2x$, $u_y = -6y$.

Therefore, LHS = $2x^2 - 6y^2 = 2(x^2 - 3y^2)$ = RHS

The adding-up theorem states that the product will be exhausted if factors are paid according to the marginal productivity theory. i.e. if $q = f(a,b)$ where q is output and a and b are inputs, then $q = af_a + bf_b$ where f_a and f_b will be the marginal products of a and b.

Let us see how it works for a linear homogeneous function.

The Cobb – Douglas is such a function defined by

$P = bL^kC^{1-K} = f(L,C)$

The adding-up theorem gives

$Lf_L + Cf_c = P$

LHS = $L(bKL^{K-1}C^{1-K}) + C(b(1-K)L^KC^{-K}) = bL^kC^{1-K}$ = RHS

This suggests that average costs and, hence, marginal costs are constant.

Partial Elasticity

Let $q_a = u(P_a, P_b)$ where q_a is the quantity of good A demanded, P_a is its price, and P_b is the price of good B.

Price elasticity is defined as $\eta = -\frac{\partial q_a}{\partial P_a}\frac{P_a}{q_a}$

This is considered as the partial elasticity of good A w.r.t P_a.

Similarly, the partial elasticity of good A w.r.t P_b is given

$$\eta = -\frac{\partial q_a}{\partial P_b}\frac{P_b}{q_a}$$

Example: $q_a = 50 - P_a - 4P_b$

Then, elasticity w.r.t. P_a is $\eta = -\frac{\partial q_a}{\partial P_a}\frac{P_a}{q_a}$. We have $\frac{\partial q_a}{\partial P_a}$ = -5. So,

$$\eta = -(-5)\,\frac{P_a}{50 - 5P_a - 4P_b}$$

For P_a = 5 and P_b = 5, $\eta = 5\,\frac{5}{50-25-20}$ = 5.

Summary

- $\frac{\partial u}{\partial x}$ is the change in u due to a small change in x keeping y constant.
- $\frac{dx}{dt}$ denotes the change in x due to a small unit change in t.
- $\frac{\partial u}{\partial x}\frac{dx}{dt}$ will be the amount of change in u due to a small change in t that is transmitted through x.
- $\frac{\partial u}{\partial y}\frac{dy}{dt}$ will be the amount of change in u due to a small change in t that is transmitted through y.
- The change in u due to a small change in t will be linear sum of these two effects and it is called the total derivative of u with respect to t.
- For u = f(x,y), x = g(t), y = h(t) where t is the independent variable. The second order total differentiation is given by

$$\frac{d^2u}{dt^2} = u_{xx}\left(\frac{dx}{dt}\right)^2 + u_{yy}\left(\frac{dy}{dt}\right)^2 + u_x\frac{d^2x}{dt^2} + u_y\frac{d^2y}{dt^2} + 2u_{xy}\frac{dx}{dt}\frac{dy}{dt}.$$

- A function is said to be homogeneous function of degree n if $f(tx,ty) = t^n f(x,y)$.
- Euler's Theorem: Let u = u(x,y) be a homogeneous function of degree n. Then

$xu_x + yu_y = nu$

- Let $q_a = u(P_a, P_b)$ where q_a is the quantity of good A demanded, P_a is its price, and P_b is the price of good B.

Price elasticity is defined as $\eta = -\frac{\partial q_a}{\partial P_a}\frac{P_a}{q_a}$.