[Academic Script]

Introduction to Partial Differentiation

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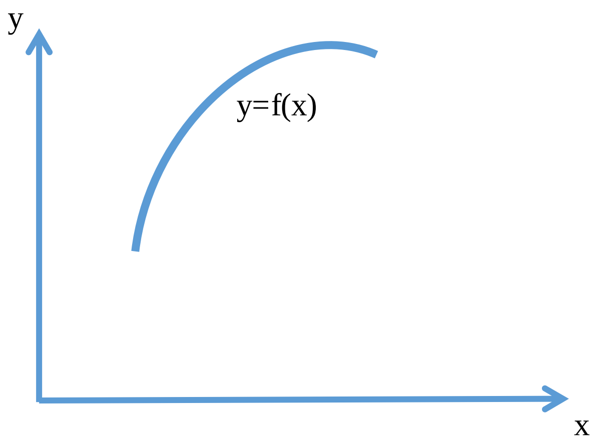
Unit No. & Title:	Unit – 2 Function of Two Variables
Lecture No. & Title:	Lecture – 1 Introduction to Partial Differentiation

Academic Script

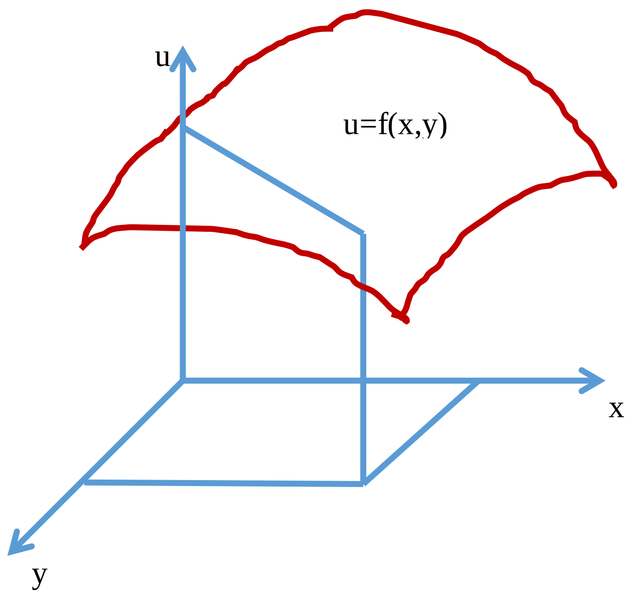
Introduction to Partial Differentiation

Functions of two variables

We have studied function of one variable y = f(x) as a mapping from one space S_1 (domain) into another space S_2 (range). In a Cartesian graph, S_1 denotes the horizontal (called x) axis and S_2 is vertical (called y) axis. The function (rule) is shown by the curve as



Now, set S_1 as a two-dimensional Cartesian space. Then we have a plane as shown in fig.

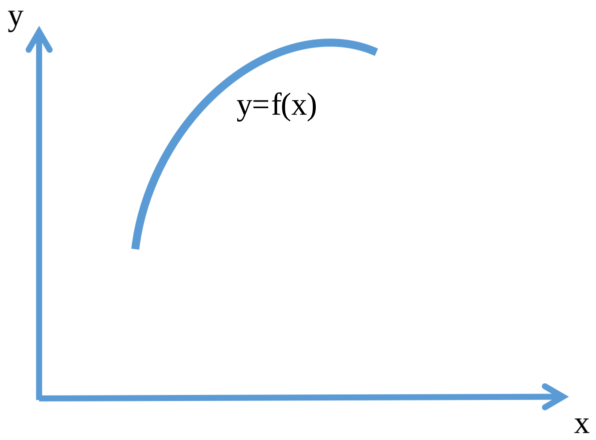


Each point on this plane is an ordered pair and is mapped into the u – axis. This function is the surface u = f(x,y).

For example, let x be the student, y be the department, and u be the university. The order pairs of x and y on the xy – plane denote the various combination of students and department. Let P be a point on this surface. Then u = f(x,y), surface maps this point into the u – axis and shows how many students will study by the combination of x and y. This surface may be regarded as total student surface.

Another example would be one in which x is vegetable, y is bread and u is utility. Then u = f(x,y) is a utility surface which shows how the various x, y combinations map into the utility axis.

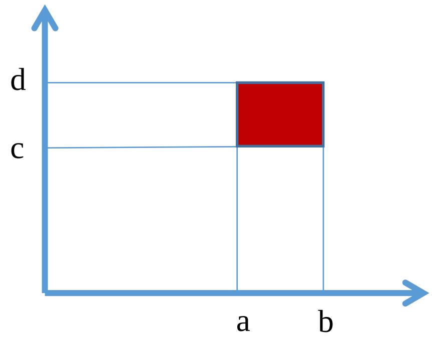
When y = c, constant, the fig. is



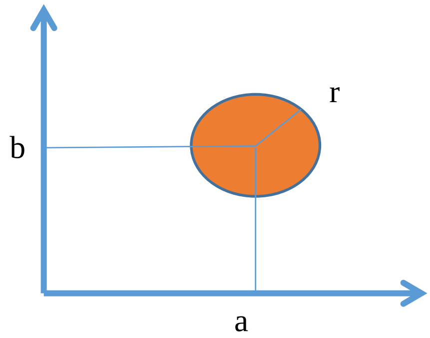
We have set up a plane passing through y = c which results into a curve AB as the intersection of plane and surface as shown in 2-D. This curve is the graph of the function u = f(x,c) which maps the point along the line c on the xy – plane into the u – axis.

The domain S_1 is called the region and the points within it are called argument points. A rectangular region can be expressed as

$$a \le x \le b, \quad c \le y \le d$$



The circular region can be expressed as $(x - a)^2 + (y - b)^2 \le r^2$ as



Partial Derivatives

Let u depends on two variables x and y as u = f(x,y).

What will be the change in u when there is a small change in x keeping y constant? As seen earlier, now u becomes a function of x only. Then derivative in this case is defined by

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, c) - f(x,c)}{\Delta x}$$

This derivative is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x.

Note: The letter ∂ (delta) shows that the other variables in the function are treated as a constant. This derivation is called the *partial derivative* of f(x,y) with respect to x.

The technique of differentiation to obtain the derivative with respect to one variable by treating other variables as constant is known as *partial derivative*.

Example: Let $f(x,y) = x^2 + 4xy + y^2$

Then $$f_x = \frac{\partial f}{\partial x} = 2x + 4y$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4$$

Note: Here f_{xy} means that the function is differentiated with respect to x first and then with respect to y.

Note: The order of the differentiation is read from right to left.

Young's Theorem: For a function $u = f(x_1, x_2, \ldots, x_n)$ with continuous first and second – order partial derivatives, the order of differentiation in computing the cross-partials is irrelevant. i.e. $f_{ij} = f_{ji}$ for $i, j = 1, 2, \ldots n$ and $i \neq j$.

Note: The statement is trivial when $i = j$.

Example: Verify Young's theorem for $f(x,y,z)= x^2e^{(3y+xz)} + 2y^3x^{-1}$.

We verify $f_{yz} = f_{zy}$.

$$f_2 = f_y$$

$$f_3 = f_z$$

$$f_{23} = f_{yz}$$

$$f_{32} = f_{zy}$$

We have $f_{yz} = f_{zy}$. So Young's theorem holds.

Marginal – Product Function

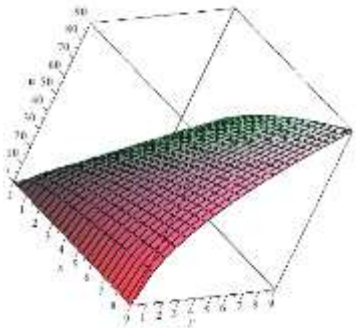
Let $u = f(x,y)$ be a production function with x and y being the levels of input and u being the level of output. Then the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are the marginal product of input x and y. This is the rate at which output increases as a result of increasing input x (or y) when there is no change in the level of other input. That is, $\frac{\partial u}{\partial x}$ denotes the change in output resulting from a one unit increase in the input x only if the choice of unit is sufficiently small.

Since the partial derivative $\frac{\partial u}{\partial x}$ will depend on the value of the other variable y, the marginal-product function of one input will be a function of all the inputs (if u is a function of n-variables).

Example: Find and interpret the partial derivatives of the production function $f(x,y) = 10x^{1/2}y^{1/2}$.

We have $f_x = \frac{5y^{1/2}}{x^{1/2}}$ and $f_y = \frac{5x^{1/2}}{y^{1/2}}$.

If we consider x as the input labor and y as the input capital, then we observe from f_x, the marginal-product of labor function, that higher values of capital leads to a more increase in output being generated by a given increase in labor. The graph is as follows.



In Fig., we see how value of y affects the derivative function f_x. The values of y = 4 and y = 9 are taken.

It suggests that for a fixed amount of capital available, the marginal-product of labor decreases as more labor is used.

Example: Find and interpret the partial derivatives for the Cobb-Douglas production function with two inputs;

$$f(K,L) = AK^aL^b,\ A > 0,\ 0 < a,\ b < 1$$

where A, a and b are technological parameters.

The marginal-product functions are

$$f_K = aAK^{a-1}L^b$$

$$f_L = bAK^aL^{b-1}$$

As a, b $\in$ (0, 1), it is observed that both the inputs satisfy the law of diminishing marginal-productivity of each input is positively related to the other output. i.e. both the inputs are complementary to each other. E.g. increasing K leads to a higher marginal product of L.

Note: The partial derivatives of f(x,y) at the point (a,b) are restricted in their direction. f_x gives a derivative parallel to x – axis and f_y gives a direction parallel to y – axis.

Differentials, Total Differential

We have seen that the partial derivatives give us a small change in u = f(x,y) when there is a small unit change in x holding y constant, or vice versa.

The total differential gives us a linear approximation of the small change in u = f(x,y) when there is a small change in both x and y.

Let u be wheat, x be labor and y be land. Then u = f(x,y) denotes wheat production by labors on the land.

We want to know what is the change in u (wheat) when there is a small change in x (labor) and in y (land).

We have seen that $\frac{\partial u}{\partial x}$ denotes the small change in u when there is a small unit change in x keeping y constant. Thus, if Δx denotes change in x then corresponding change in u is $\frac{\partial u}{\partial x}\,\Delta x$. Similarly, when change in y is Δy, then the change in u is $\frac{\partial u}{\partial y}\,\Delta y$.

As a first approximation, the change in u due to a small change in x and y is

$$\Delta u = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y$$

Let du = Δu, dx = Δx and dy = Δy. Then

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$du = f_x dx + f_y dy$$

Example: Given $u = f(x,y) = 3x^2 + 4y^2$.

The total differential is $du = f_x dx + f_y dy$

We have $f_x = 6x$ and $f_y = 8y$. So, $du = 6xdx + 8ydy$.

Example: Let u be utility and x and y be goods. Then the utility function is $u = f(x,y)$.

When there is a small change in x and y, the corresponding change in utility is $du = f_x dx + f_y dy$ where f_x is the marginal utility with respect to good x and f_y is the marginal utility with respect to good y.

Rules of Differentiations

Rule 1. Differentiation of a sum

$$u = f + g \text{ then } du = df + dg.$$

Rule 2. Differentiation of a product

$$u = fg \text{ then } du = fdg + gdf$$

Example: $u = (x^2 + 2y)(x+y^2)$

We have $du = (x^2 + 2y)d(x+y^2) + (x+y^2)d(x^2 + 2y)$

$$= (x^2 + 2y)(dx + 2ydy) + (x+y^2)(2xdx + 2dy)$$

$$= (3x^2 + 2y + 2xy^2)dx + (2x^2y+6y^2+2x+2xy^2)dy$$

Rule 3. Differentiation of a quotient

$$u = \frac{f}{g} \text{ then } du = \frac{gdf - fdg}{g^2}$$

Example: Given $u = \frac{x^2+y^2}{x+y}$.

We have $du = \frac{(x+y)d(x^2+y^2)-(x^2+y^2)d(x+y)}{(x+y)^2}$

$$= \frac{(x+y)(2xdx+2ydy)-(x^2+y^2)(dx+dy)}{(x+y)^2}$$

$$= \frac{(x^2-y^2+2xy)dx-(x^2-y^2-2xy)dy}{(x+y)^2}$$

Rule 4. Chain Rule: Let $z = z(u)$ and $u = u(x)$. The differential of z is $dz = d(z(u)) = \frac{d}{dx}(z)du$.

Here, du is differential of u and it is given by

$$du = d(u(x)) = \frac{d}{dx}(u)dx$$

Hence, $dz = \left[\frac{d}{du}(z)\right]\left[\frac{d}{dx}(u)dx\right]$.

Example: Given $z = u^2 + 1$, $u = x^2 + 2$. Compute dz.

$$dz = \left[\frac{d}{du}(u^2 + 1)\right]\left[\frac{d}{dx}(x^2 + 2)dx\right]$$

$$= (2u)(2x)dx = 4uxdx = 4(x^2+2)dx$$

Example: Given $z = u^2 + 1$, $u = x^2 + y^2$. Compute dz.

$$dz = 2udu$$

$$du = d(x^2 + y^2) = 2xdx + 2ydy$$

Therefore, $dz = 2u(2xdx + 2ydy)$

$$= 2(x^2 + y^2)(2xdx + 2ydy)$$

$$= 4(x^3 + xy^2)dx + 4(x^2y + y^3)dy$$

Rule 5. Differential of e^x

$$dy = d(e^x)dx = e^x dx$$

Example: Given $y = e^{x^2+1}$. Compute dy.

$$dy = d(e^{x^2+1}) = e^{x^2+1}\, d(x^2 + 1)dx$$

$$= e^{x^2+1}\, 2xdx = 2xe^{x^2+1}dx$$

Example: Given $z = e^{x^2+y^2}$. Compute dz.

Let $u = x^2 + y^2$. Then $dz = \frac{d}{du}\, e^u du$ and

$$du = d(x^2 + y^2) = 2xdx + 2ydy$$

So, $dz = e^{x^2+y^2}(2xdx + 2ydy)$.

Rule 6. Differential of $y = \log x$.

$$dy = d(log x) = \left(\frac{d}{dx}\, log x\right)dx = \frac{1}{x}\, dx$$

Example: Given $z = \log(x^2 + y^2)$. Compute dz.

Let $u = x^2 + y^2$. Then $dz = \frac{d}{du}\log u\, du = \frac{1}{u}du = \frac{1}{x^2+y^2}du$ and

$$du = d(x^2 + y^2) = 2xdx + 2ydy$$

Then $dz = \frac{2x}{x^2+y^2}dx + \frac{2y}{x^2+y^2}dy$.

Second and Higher-order differentials for Two-Variable case: With independent variables:

Consider u = f(x,y)

where x and y are independent variables. Then the first order differential du (df) is

$$df = f_x dx + f_y dy$$

The second order differential is

$d^2f = d(df) = \frac{\partial}{\partial x}(df)dx + \frac{\partial}{\partial y}(df)dy$

$= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy$

$= [f_{xx}dx + f_x\frac{\partial}{\partial x}(dx) + f_{xy}dy + f_y\frac{\partial}{\partial x}(dy)]dx$

$+ [f_{xy}dx + f_x\frac{\partial}{\partial y}(dx) + f_{yy}dy + f_y\frac{\partial}{\partial y}(dy)]dy$

But, since dx and dy are considered as constants,

$\frac{\partial}{\partial x}(dx) = 0, \frac{\partial}{\partial x}(dy) = 0, \frac{\partial}{\partial y}(dx) = 0, \frac{\partial}{\partial y}(dy) = 0$

Thus, $d^2f = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = (f_x dx + f_y dy)^2$.

In general, $d^nf = (f_x dx + f_y dy)^n$.

For example,

$d^3f = (f_x dx + f_y dy)^3 = f_{xxx}dx^3 + 3f_{xxy}dx^2dy + 3f_{xyy}dxdy^2 + f_{yyy}dy^3$

Example: Find the second-order differential of $f = x^2 + xy$.

We have

$$d^2f = f_{xx}(dx)^2 + 2f_{xy}dxdy + f_{yy}(dy)^2$$

$$f_x = 2x + y\ ,\ f_y = x$$

$$f_{xx} = 2\ ,\ f_{xy} = 1,\ f_{yx} = 1,\ f_{yy} = 0$$

Thus, we get

$$df = (2x + y)dx + xdy$$

$$d^2f = 2(dx)^2 + 0(dy)^2 + 2(1)dxdy = 2(dx)^2 + 2dxdy$$

Note: For a function of three variables. u = f(x,y,z), n-th order differential is given by

$$(d^nf) = (f_x dx + f_y dy + f_z dz)^n$$

One-variable case: dependent variables

Consider $y = f(u)$, $u = F(x)$

where u is a function of x. Then

$$dy = d(f(u)) = f'(u)du$$

and du is not an independent variable.

The second-order differential of y is

$$d^2y = d(dy) = d[f'(u)du] = f'(u)d^2u + f''(u)(du)^2$$

where d^2u is the second-order differential of u and given by $d^2u = F''(x)(dx)^2$.

When u is an independent variable and does not depend on x, $d^2u = 0$. Thus, $d^2y = f''(u)(du)^2$.

Example: Find d^2y when $y = u + u^2$ and $u = x^2 + 1$.

We have $dy = f'(u)du$

and $d^2y = f'(u)d^2u + f''(u)(du)^2$.

$f'(u) = 1 + 2u$, $f''(u) = 2$.

$du = 2xdx$, $d^2u = 2(dx)^2$.

Therefore, $d^2y = (1+2u)2(dx)^2 + 2(2xdx)^2 = (12x^2 + 6)(dx)^2$

Two-variable case – dependent variables

Given $u = f(x,y)$ and $\phi(x,y) = 0$

where x and y are not independent variables. Then

$$d^2u = f_x d^2x + f_{xx}(dx)^2 + f_{xy}(dy)^2 + 2f_{xy}dxdy$$

Example: Given a utility function $u = q_1q_2$ and budget constraint $M = p_1q_1 + p_2q_2$. Find d^2u.

We can write $q_1 = \frac{M-p_2q_2}{p_1}$.

$dq_1 = \frac{d}{dq_2}\left(\frac{M-p_2q_2}{p_1}\right)dq_2 = -\frac{p_2}{p_1}dq_2$, $d^2q_1 = 0$

$u_{q_1} = q_2$, $u_{q_1q_1} = 0$, $u_{q_1q_2} = 1$, $u_{q_2} = q_1$, $u_{q_2q_2} = 0$, .

Hence,

$d^2u = u_{q_1}d^2q_1 + u_{q_1q_1}(dq_1)^2 + u_{q_2q_2}(dq_2)^2 + 2u_{q_1q_2}dq_1q_2 = 2dq_1dq_2$

SUMMARY

- A function of several variables consists of two parts: a domain, which is a collection of points in the plane or in the space, and a rule, which assigns to each member of the domain one and only one point.
- A function of several variables is called a function of two variables if its domain is a set of points in the plane.
- A function of several variables is called a function of three variables if its domain is a set of points in the space.
- Let u = f(x,y). The derivative of u with respect to x if it exists when x alone varies and y remains constant is called partial derivative of u with respect to x. and it is denoted by u_x.
- Let u = f(x,y). The derivative of u with respect to y if it exists when y alone varies and x remains constant is called partial derivative of u with respect to y. and it is denoted by u_y.
- **Young's Theorem**: For a function $u = f(x_1, x_2, \ldots, x_n)$ with continuous first and second – order partial derivatives, the order of differentiation in computing the cross-partials is irrelevant. i.e. $f_{ij} = f_{ji}$ for i, j = 1, 2,…n and i ≠ j.
- Let u = f(x,y) be a production function with x and y being the levels of input and u being the level of output. Then the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are the marginal product of input x and y. This is the rate at which output increases as a result of increasing input x (or y) when there is no change in the level of other input. That is, $\frac{\partial u}{\partial x}$ denotes the change in output resulting from a one-unit increase in the input x only if the choice of unit is sufficiently small.
- Differentiation of a sum

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- Differentiation of a product

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- Differentiation of a quotient

$$u = \frac{f}{g} \text{ then } du = \frac{gdf - fdg}{g^2}$$

- Chain Rule: Let z = z(u) and u = u(x). The differential of z is dz = d(z(u)) = $\frac{d}{dx}(z)du$.