[Academic Script]

[Elementary Difference Equations &Their Applications to Economics]

Subject:	Business Economics
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Unit No. & Title:	Unit - 4 Difference Equations
Lecture No. & Title:	1: Elementary Difference

1. Introduction to Differential Equations:

Suppose there is a functional relational between x and y, which can be represented as- $y = x^2$ $0 \leq x \leq 3$ ------------------------------------- (1)

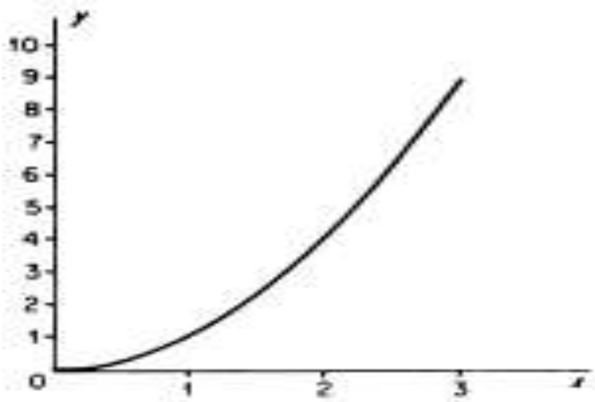
in this equation x is the independent variable and y is the dependent variable.

The derivative of y will be

$\frac{\mathrm{dy}}{\mathrm{dx}} = 2x$ ---------------------------------- (2)

Thus, for $x = 2, dy/dx = 4$, which is the slope of the curve at the point (2,4).

For $x = 3, dy/dx = 6$, which is the slope of the curve at the point (3,9).



Now suppose we have an equation such as (2) which directly assumes a functional relation between x and y and includes x, y and the derivative of y, with respect to x. Such an equation is called a *differential equation*. So we can say that, a differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

Differential equations can be broadly classified into ordinary differential equations and partial differential equations.

The *ordinary differential equations* are those having only one independent variable and its derivatives.

When the differential equation has several independent variables then the derivatives become partial derivatives and we have *partial differential equations*.

An ordinary differential equation is said to be linear if the dependent variables is of the first degree; i.e., the y-variable and its derivatives are of the first power only. In this case the y-variable and its derivatives are linearly combined.

<u>The general form of a linear differential equation is</u>-

$$h_0(x)\frac{d^n y}{dx^n} + h_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + h_{n-1}(x)\frac{dy}{dx} + h_n(x)y = f(x)$$

This is an ordinary linear differential equation of order n.

If the coefficients of y and its derivatives, i.e., $h_0(x), h_1(x),\dots\ h_n(x)$ are all constants, then it is called a linear differential equation with constant coefficients.

Let $a_0, a_1 \ldots\ldots a_n$ be constants, then

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x)$$

If $f(x) = 0$, then we have a homogeneous linear equation with constant coefficients

1.1 Classification of Differential Equations:

- Non-linear differential equations of the first order and first degree.
    a) Variables are separable case
    b) Homogeneous differential equation
    c) Exact differential equation
- Linear differential equations of the first order
    a) Homogeneous differential equation with constant coefficients
    b) Non- Homogeneous differential equation with constant coefficients
    c) General case
- Linear differential equations of the second order with constant coefficients.

1.1.1. Non-linear differential equations of the first order and first degree

There are some special types of non-linear differential equations that can be solved , which are of the first order and first degree, for instance–

$$\frac{\mathrm{dy}}{\mathrm{dx}} = f(x,y)$$

We can understand this type of equation with an example –

$$\frac{\mathrm{dy}}{\mathrm{dx}} = (3xy^2)$$

In differential form it is written as-

$$f_1(x,y)dx + f_2(x,y)dy = 0$$

<u>Case 1: when variables are separable</u>

In the differential equation

$$\frac{\mathrm{dy}}{\mathrm{dx}} = f(x,y)$$

We can put the above equation in the form-

$$f_1(x)dx + f_2(y)dy = 0$$

This is easy integrals because we can associate a function $f_1(x)$ with dx which is only a function of x and with dy we associate a function $f_2(y)$ which is only function of y. This is the case of variables separable.

<u>Case II: differential equation with homogeneous coefficients:</u>

If $f(x,y)$ is a homogeneous function of degree n, then

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

If the differential equation

$$f_1(x,y)dx + f_2(x,y)dy = 0$$

has homogeneous coefficients, that is if $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of the same order, then by change of variable we can reduce it to the variables separable case.

Let f_1 and f_2 be of order n; i.e., of the same order.

Now suppose-

$y = vx,$

$dy = vdx + xdv$

Then,

$$f_1(x,vx)dx + f_2(x,vx)(vdx + xdv) = 0$$

$$\frac{f_1(x,vx)}{f_2(x,vx)}dx + vdx + xdv = 0$$

Since f_1 and f_2 are homogeneous functions we can divide out the x. Thus,

$$\left(\frac{f_1(1,v)}{f_2(1,v)} + v\right)dx + xdv = 0$$

And hence we have the variables separable case which can be solved by integration.

Case III: exact differential equations

Consider a function-

$f(x,y) = x^2y$

The differential of this function is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= 2xy\,dx + x^2dy$$

Assume :

$P(x,y) = 2xy$ $\qquad$ $Q(x,y) = x^2$

As we know,

$$\frac{\partial P}{\partial y} = f_{xy} = 2x = f_{yx} = \frac{\partial Q}{\partial x}$$

Now we reverse the procedure. We are given an expression-

$2xy\,dx + x^2dy$ $\qquad$ Or

$P(x,y)dx + Q(x,y)dy \ldots \ldots \ldots (1)$

If the following conditions hold by the expression then this expression is said to be the exact differential of some function $f(x, y)$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ……………………….(2)

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x$$

Thus, the condition (2) holds, and we know (1) is an exact differential of some function $f(x, y)$.

Let us now set (1) equal to zero.

$P(x, y)dx + Q(x, y)dy = 0$………………(3)

Hence, $2xy\, dx + x^2 dy = 0$

This is called an exact differential equation.

1.1.2. <u>Linear differential equations of the first order</u>

Let us now discuss about linear differential equations.

The general expression is-

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x or constants. The general solution for this equation is given by –

$y = e^{-\int P dx} \int e^{\int P dx} Q\, dx + ce^{-\int P dx}$…………..(1)

Case 1: homogeneous differential equation:

When Q=0, and P is a function of x, we have a homogeneous differential equation and the solution is,

$y = ce^{-\int Pdx}$……………………………………..(2) (from 1)

Case 2: Non homogeneous differential equation

When P and Q are constants and we have a non-homogeneous differential equation with constant coefficients, the solution becomes

$$y = e^{-a1x}\int e^{a1x}a_2dx + ce^{-a1x}$$

$$= ce^{-a1x} + \frac{a_2}{a_1}$$

Case 3: general case:

When P and Q are functions of x and we have the general case, the solution is-

$$\frac{dy}{dx} + \frac{1}{x}y = x$$

$$y = e^{-1nx}\int e^{1nx}x\,dx + ce^{-1nx}$$

$$= \frac{x^2}{3} + \frac{c}{x}$$

1.1.3. Linear differential equation of second order with constant coefficient

This type of equation is expressed by-

$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x)$……..(1)

This solution is obtained in two steps-

Step 1 is to set $f(x) = 0$ and solve

$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$……….(2)

This is called a homogeneous equation, and the solution is called the complementary function for equation (1)

Let this be y_c

Step 2 is to find a particular solution of equation (1).

Let this be y_v

Then the general solution y will be-

$$y = complementary\ function + particular\ function$$

$$= y_c + y_v$$

1.2. <u>Indefinite Integrals:</u>

The problem of computing areas under the graph of a function f leads to the problem of finding an antiderivative of f- that is, a function F whose derivative is f.

We follow the usual practice and call F an indefinite integral of f. We use symbol for an indefinite integral of f, we use $\int f(x)dx$. If two functions have the same derivative throughout an interval then it must be differed by a constant, so we can write-

$\int f(x)dx = F(x) + C$	When	$F'(x) = f(x)$…….(1)

For example:

$\int x^3 dx = \frac{1}{4}x^4 + C$ because $\left(\frac{1}{4}x^4\right)' = x^3$

Where ( )' denotes differentiation. The symbol $\int$ is the integral sign, the function $f(x)$ appearing in (1) is the integrand, and C is the constant of integration. The dx part of the integral notation indicates that x is the variable of integration.

Let a be a fixed number ≠ -1. Because the derivative of $x^{a+1}/(a+1)$ is x^a.

$$\boxed{\int x^a dx = \frac{1}{a+1}x^{a+1} + C \qquad (a \neq -1)}$$

This very important integration result states that the indefinite integral of any power of x (except x^{-1}) is obtained by increasing the exponent of x by 1, dividing by the new exponent and then adding the constant of integration.

Some general rules:

Two rules of differentiation are $\left(a\,F\,(x)\right)' = aF'(x)$ and

$\left(F(x) + G(x)\right)' = F'(x) + G'(x)$. They immediately imply the following integration rules:

Constant Multiple Property :

$\int af(x)dx = a\int f(x)dx$ (a is a real constant)

The integral of a sum is the sum of the integrals:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Repeated use of the above properties yields the general rule

$$\int [a_1 f_1(x) + \cdots + a_n f_n(x)]dx = a_1 \int f_1(x)dx + \cdots + a_n \int f_n(x)dx$$

For the indefinite integral of any linear combination of continuous function.

1.3. The Definite Integral:∈

Let f be a continuous function defined in the interval $[a, b]$. Suppose that the function F is continuous in $[a, b]$ and has a derivative satisfying $F'(x) = f(x)$ for every $x \epsilon (a, b)$. Then the difference $F(b) - F(a)$ is called the definite integral of f over $[a, b]$. This difference does not depend on which of the infinitely many indefinite integrals of f we choose as *F*. the definite integral of f over $[a, b]$ is therefore a number that depends only on the function f and the numbers a and b. We denote it by

$$\int_a^b f(x)dx$$

This notation makes explicit the function $f(x)$ we integrate, which is called the integrand, and the interval of integration $[a, b]$. The numbers a and b are called, respectively, the lower and upper limits of integration. The letter x is a dummy variable in the sense that the integral is independent of its label.

For instance,

$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(\xi)d(\xi)$

In many other mathematical writings, the difference $F(b) - F(a)$ is often denoted by $F(x)\Big|_a^b$, or by $[F(x)]_a^b$. But $\Big|_a^b F(x)$ is also common, and this is the notation we shall use. Thus:

Definition of definite integral-

$$\boxed{\int_a^b f(x)dx = \Big|_a^b F(x) = F(\ \ ) - F(a)}$$

Where $F'(x) = f(x)$ for all $x \in (a, b)$.

The definition does not require $a < b$. However, if $a > b$ and $f(x)$ is positive throughout the interval $[b, a]$, then $\int_a^b f(x)dx$ is a negative number

Properties of the Definite Integral:

From the definition of the definite integral in the box, a number of properties can be derived. If f is a continuous function in an interval that contains a, b and c, then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_b^a f(x)dx = 0$$

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

<u>Economic Application of Integration:</u>

The integral has many important interpretations. For example, we are led to a definite integral when we want to find the volume of a solid of revolution or the length of a curve. Several of the most important concepts in statistics are also expressed by integrals of continuous probability distributions.

Some of the examples are discussed below which shows the importance of integrals-

- <u>Extraction from an oil well:</u>

  Assume that at time t=0, we start extracting oil from a well that contains K barrels of oil. Let us define

  $$x(t) = amount\ of\ oil\ in\ barrels\ that\ is\ left\ at\ time\ t$$

  In particular, $x(0) = K$. If we assume that we cannot pump oil back into the well, then $x(t)$ is a decreasing function of t. The amount of oil that is extracted in a time interval $[t. t + \Delta t]$ (where Δt> 0) is $x(t) - x(t + \Delta t)$. Extraction per unit of time is, therefore,

  $$\frac{x(t) - x(t + \Delta t)}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

If we assume that $x(t)$ is differentiable, then the limit as Δt approaches zero of the fraction (*) is equal to $-x(t)$. Letting $u(t)$ denote the rate of extraction at time t, we have

$\dot{x}(t) = -u(t)$ with $x(0) = K$

The solution to the initial – value problem is

$$x(t) = K - \int_0^t u(\tau)d\tau$$

If we check the above equation-

First, setting $t = 0$ gives $x(0) = K$. Moreover, differentiating the equation yields $\dot{x}(t) = -u(t)$. The result may be interpreted as follows:

The amount of oil left at time t is equal to the initial amount K, minus the total amount that has been extracted during the time span $[0, t]$, namely $\int_0^t u(\tau)d\tau$.

If the rate of extraction is constant, with $u(t) = \bar{u}$, then the result will be-

$x(t) = K - \int_0^t \bar{u}d\tau = K - \Big|_0^t \bar{u}\,\tau = K - \bar{u}t$

In particular, we see that the well will be empty when $K - \bar{u}t = 0$, or when $t = K/\bar{u}$.

- <u>A country's foreign exchange reserves:</u>

Let $F(t)$ denote a country's foreign exchange reserves at time t. Assuming that F is differentiable, the rate of change in the foreign exchange reserves per unit of time will be-

$$f(t) = F'(t)$$

If $f(t) > 0$, this means that there is a net flow of foreign exchange into the country at time t, whereas $f(t) < 0$ means that foreign exchange is flowing out. From the definition of the definite integral, it follows that

$$F(t_1) - F(t_0) = \int_{t_0}^{t_1} f(t)dt$$

The above equation measures the change in the foreign exchange reserves over the time interval$[t_0 t_1]$.

Summary:

In this session, we have learnt about the meaning of differential equations along with the meaning and forms of ordinary differential and partial differential equations. Moreover, different classification of differential equations and economic applications of integration has also been discussed.