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Simultaneous Equations Model

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1. Introduction

Hello friends nice meeting you. When we deal with econometric applications usually there is only one equations and number of variables. Many times it happens that there are so many equations and a number of variables including those equations. A speciality is that one variable which is dependent in one equation can occur as an independent variable in another equation and vice versa. Due to this is it possible to estimate a single equation. There are certain things which are known as identification. There are certain rules of Identification, order and rank condition and first of all we have to apply them and then we can think of estimation. Usual OLS estimation will fail because they will give inconsistent estimators. So new methods are to be devised all this things you will be learn in this lecture.

In these models, there are more than one equations and more than one dependent variables. In Single equation models there is only one equation relating a single dependent variable to a set of explanatory variables which are either non-stochastic or if stochastic they are assumed to be independently distributed with the stochastic disturbance term.

In the case of simultaneous equations system model the speciality is that a dependent variable in one equation may appear as an explanatory variable in another equation of the system. Due to this reason, such a dependent variable becomes stochastic and it is usually correlated with the disturbance term of the equation in which it appears as an explanatory variable.

(**Illustration**

Consider the following model

$Y_t = C_t + I_t$... (i)

$C_t = \beta_0 + \beta_1 Y_t + U_t$............................ (ii)

$I_t = \alpha_0 + \alpha_1(Y_t - Y_{t-1}) + \vartheta_t$........................ (iii)

Y=Income, C=Consumption, I= Investment, t=time.

The above 3 equations represent a simultaneous equations system model. Here Y_t in equation (i) appears as dependent variable, but it occurs as explanatory variable in equation (ii). Also C_t in equation (ii) is dependent variable and it appears in equation (i) as explanatory variable. Similarly I_t in equation (i) appears as explanatory variable but it appears as dependent variable in equation (iii) and so on. Thus it may not be proper to take any equation (equation (ii) or (iii)) separately to run regression since all the equations are shown related in the model as shown.)

If we use classical least squares method then the estimators that we obtain will be biased and this bias is not removed even if we use large samples, that are the estimators are inconsistent. We shall show this by means of a theorem illustrated with the help of two variables model.

Theorem OLS estimation for simultaneous equations system model leads to biased and inconsistent estimators.

Proof: We shall establish this result by using 2 variables model.

Let us consider the following model

$C_t = \beta_0 + \beta_1 Y_t + U_t \rightarrow (1)$

$Y_t = C_t + I_t \qquad \rightarrow (2)$

Where C_t= Consumption at time t

Y_t = Income at time t

And I_t= Investment at time t

Here $0 < \beta_1 < 1$

We assume that and $E(u_t) = 0, V(u_t) = E(U_t^2) = \sigma^2$

$E(U_t, U_{t+j}) = 0 \ (j \neq 0) \ and \ Cov(I_t, U_t) = 0$

We shall show that Y_t and U_t are correlated and also prove that $\hat{\beta}_1$ obtained by OLS is a biased and inconsistent estimator.

From (1) and (2) we get

$Y_t = \beta_0 + \beta_1 Y_t + I_t + U_t$

Hence $Y_t = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t + \frac{1}{1-\beta_1} U_t \qquad (\beta_1 \neq 1)$

So that $E(Y_t) = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t$

Which gives $Y_t - E(Y_t) = \frac{U_t}{1-\beta_1}$

Now $U_t - E(U_t) = U_t$

$\therefore Cov(Y_t, U_t) = E(\{Y_t - E(Y_t)\}\{U_t - E(U_t)\}) = \frac{\sigma^2}{1-\beta_1}$

which shows that Y_t and U_t are correlated, which violates the basic assumption of classical linear model

Now if we use OLS method then using relation (1)

We get $\hat{\beta}_1 = \frac{\sum_t (C_t - \bar{c})(Y_t - \bar{Y})}{\sum_t (Y_t - \bar{Y})^2}$

Let us write $(C_t - \bar{c}) = c_t$ and $(Y_t - \bar{Y}) = y_t$

Then $\hat{\beta}_1 = \frac{\sum c_t y_t}{\sum y_t^2}$

Since

$C_t = \beta_0 + \beta_1 Y_t + U_t$

We get $\sum c_t y_t = \sum(\beta_0 + \beta_1 Y_t + U_t) y_t / \sum y_t^2$

Hence $\hat{\beta}_1 = \beta_1 + \frac{\sum y_t U_t}{\sum y_t^2}$

From which $E(\hat{\beta}_1) = \beta_1 + E\left[\frac{\sum y_t U_t}{\sum y_t^2}\right]$

Hence OLS of $\hat{\beta}_1$ is a biased estimator for β_1.

We wish to know whether $\hat{\beta}_1$ is consistent for β_1

[By definition an estimator T_n for parameter θ is called consistent if its probability limit is equal to θ, that if is consistent for θ if plim(T_n)=θ]

Hence $\hat{\beta}_1$ will be consistent for β_1 if plim $(\hat{\beta}_1) = \beta_1$

We have plim $(\hat{\beta}_1) = plim(\beta_1) + plim\left(\frac{\sum U_t y_t}{\sum y_t^2}\right)$

$$= \beta_1 + \frac{plim(\sum U_t y_t)}{plim(\sum y_t^2)}$$

$$= \beta_1 + \frac{plim(\sum U_t y_t/N)}{plim(\sum y_t^2/N)}$$

Let $\sum U_t y_t/N$ =sample covariance between y_t and $U_t = \delta$

$\sum y_t^2/N =$ sample variance of $Y_t = \mu$

Then plim $(\hat{\beta}_1) = \beta_1 + \frac{plim(\delta)}{plim(\mu)}$

As $N \to \infty$, plim(δ) $= \frac{\sigma^2}{(1-\beta)}$ and plim(μ) $= V(Y_t) = \sigma_Y^2$

$\therefore$ plim $(\hat{\beta}_1) = \beta_1 + \left(\frac{1}{1-\beta_1}\right)\left(\frac{\sigma^2}{\sigma_Y^2}\right)$

$\sigma^2 > 0, \sigma_Y^2 > 0, 0 < \beta_1 < 1$

Hence plim $(\hat{\beta}_1) > \beta_1$

This shows that by using OLS method, $\hat{\beta}_1$ is a biased and inconsistent estimator of β_1. Hence OLS estimation method cannot be considered to be appropriate if used directly.

2. Identification Problem

For a given simultaneous equations model the equations given in the model are called structural equations (or behavioural equations). The parameters in the structural equations are called structural parameters (or structural coefficients).

From the given model we can find reduced form of equations. A reduced form equation is one which expresses an endogenous variable completely in terms of the predetermined variables and the stochastic disturbances. The coefficient obtained in the reduced form equations are called reduced forms coefficients.

Note that reduced form coefficients are some functions of structural parameters.

Let us first illustrate this by a simple example.

Illustration-1 Let our model be expressed as under

$C_t = \beta_1 + \beta_2 Y_t + U_t \quad (0 < \beta_1 < 1)$.................... (1)

and $Y_t = C_t + I_t$... (2)

These two are called structural equations and β_1 and β_2 are the structural parameters.

Insert first equation in second one, so we get

$Y_t = \pi_1 + \pi_2 I_t + W_t$....................................... (3)

Where

$$\left.\begin{aligned} \pi_1 &= \beta_1/(1-\beta_2) \\ \pi_2 &= 1/(1-\beta_2) \\ W_t &= U_t/(1-\beta_2) \end{aligned}\right\} \text{.............................. (4)}$$

This equation is called reduced form equation and the coefficient π_1 and π_2 are called reduced forms coefficients. We can run regression for equation (3) and obtain estimators $\widehat{\pi_1}$ and $\widehat{\pi_2}$ and hence find $\hat{\beta}_1$ and $\widehat{\beta}_2$. In other words, we can also say that the parameters β_1 and β_2 of the model can be estimable or the given equation (1) is identified. From this now we turn to the concept of Identification.

Identification problem means whether numerical estimates of the parameters of a structural equation can be obtained from the estimated reduced form coefficients. If this can be done, the particular equation is identified. If this cannot be done then equation is called unidentified or under identified.

An identified equation can be said to be exactly identified is unique numerical values of the structural parameters can be obtained. An identified equation can be said to be over identified if more than one numerical values can be obtained for some of the parameters of the structural equations.

In the above illustration we have $\hat{\beta}_1 = \widehat{\pi_1}$ and hence $\hat{\beta}_2 = 1 - \frac{1}{\widehat{\pi_2}}$ which gives unique solutions for β_1 and β_2 based upon the reduced from equations. Hence the given equation (1) of the model is just or exactly identified. Next we give below an illustration for unidentification.

Illustration-2

Consider the following model

$Q_t^d = \alpha_0 + \alpha_1 P_t + U_{1t} \quad (\alpha_1 < 0) \rightarrow (1)$

$Q_t^s = \beta_0 + \beta_1 P_t + U_{2t} \quad (\beta_1 > 0) \rightarrow (2)$

$Q_t^d = Q_t^s \qquad\qquad\qquad \rightarrow (3)$

Here Q_t^d =quantity demanded, Q_t^s = quantity supplied, P_t = price

From (3), we get from (1) and (2)

$\alpha_0 + \alpha_1 P_t + U_{1t} \;= \beta_0 + \beta_1 P_t + U_{2t}$

Hence $P_t = \pi_0 + v_t \rightarrow (4)$

Where $\pi_0 = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \rightarrow (5)$, $v_t = \frac{U_{2t} - U_{1t}}{\alpha_1 - \beta_1} \rightarrow (6)$

Now $Q_t = \alpha_0 + \alpha_1(\pi_0 + v_t) + U_{1t}$

Which gives on simplification $Q_t = \pi_1 + W_t \rightarrow (7)$

Where $\pi_1 = \frac{\alpha_0\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1}$, $\rightarrow (8)$ and $W_t = \frac{\alpha_1 U_{2t} - \beta_1 U_{1t}}{\alpha_1 - \beta_1} \rightarrow (9)$

There are 4 structural parameters $\alpha_0, \alpha_1, \beta_0, \beta_1$ to be estimated. There are 2 reduced from coefficients π_0 and π_1 containing the above 4 structural parameters. We do not get any solution for obtaining the estimates of the parameters. Hence we conclude the equation in the structural modes cannot be identified.

Illustration-3

Consider the following model

Demand : $Q_t^d = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + U_{1t} \rightarrow (10)$

Supply : $Q_t^s = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + U_{2t} \qquad \rightarrow (11)$

Equilibrium condition : $Q_t^d = Q_t^s \qquad\qquad \rightarrow (12)$

Here I= Income, P= price, R= wealth

First if we solve for P_t, we get the relation

$P_t = \pi_0 + \pi_1 I_t + \pi_2 R_t + \pi_3 P_{t-1} + v_t \rightarrow (13)$

And then by substitution if we solve for Q_t we get

$Q_t = \pi_4 + \pi_5 I_t + \pi_6 R_t + \pi_7 P_{t-1} + w_t \rightarrow (14)$

v_t and w_t are stochastic disturbance terms.

Here

$$\left.\begin{array}{ll}
\pi_0 = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} & \pi_1 = -\dfrac{\alpha_2}{\alpha_1 - \beta_1} \\
\pi_2 = -\dfrac{\alpha_3}{\alpha_1 - \beta_1} & \pi_3 = \beta_2/(\alpha_1 - \beta_1) \\
\pi_4 = \dfrac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} & \pi_5 = -\dfrac{\alpha_2\beta_1}{\alpha_1 - \beta_1} \\
\pi_6 = -\dfrac{\alpha_3\beta_1}{\alpha_1 - \beta_1} & \pi_6 = \dfrac{\alpha_1\beta_2}{\alpha_1 - \beta_1} \\
W_t = \dfrac{\alpha_1 U_{2t} - \beta_1 U_{2t}}{\alpha_1 - \beta_1} & v_t = \dfrac{U_{2t} - U_{1t}}{\alpha_1 - \beta_1}
\end{array}\right\} \rightarrow (15)$$

Here we have 7 structural coefficients and 8 reduced force coefficients.

In this case we get $\beta_1 = \frac{\pi_6}{\pi_2}$ and also $\beta_1 = \frac{\pi_5}{\pi_1}$

Thus we get two solutions for β_1.

This is called over identification where we get more solutions for the parameters. We say that supply function is over identified. Also since β_1 appears in the denominator of all the above relations, other parameters can also not uniquely estimated.

2. Rules for Identification

We have seen in the above representation by means of 3 possibilities for identification

(1)Just identified equation

(2)Over Identified equation

(3)Unidentified equation

The above method of deciding about identification seems to be a crude one. Hence some systematic approach may be necessary to take decision about identification. For this we have two rules of identification (1) Order condition and (2) Rank condition.

(1) Order condition

Let us define

G = Number of equations in the given model (This also refers to the number of endogenous variables in the model)

M = Number of variables in the given equation of the model (This also refers to number of endogenous and predetermined variables in the given equation of the model)

K = Number of variables in the model (This also refers to the number of endogenous and predetermined variables in the model)

Rule (i) If $K - M > G - 1 \rightarrow$ the equation is **over identified**

(ii) If $K - M = G - 1 \rightarrow$ the equation is **just identified**

This rule can be expressed in words as under

"If the given equation of a model is to be identified then the number of endogenous and predetermined variables which are excluded from the model should be greater than or equal to the number of endogenous variables minus one".

(2) Rank condition

Let G denote the number of equations in the given model then a particular equation of the model can be said to be identified if and only if at least one non-zero determinant minor of order $(G-1) \times (G-1)$ can be constructed from the coefficients of the variables (both endogenous and predetermined) excluded from that particular equation but included in other equations of the model.

Find rank of such matrix A.

Rule (i) If $\rho(A) = G - 1 \rightarrow$The equation is **identified**

(ii) If $\rho(A) < G - 1 \rightarrow$The equation is **unidentified**

Note that (a) order condition is a necessary condition whereas rank condition is necessary and sufficient condition for identification. Both of them must be checked for the relevant equation for which we want to consider identification. (b) Rank condition stated above can be done in a systematic way and it will be illustrated by means of examples.

Example-1 Apply order and rank conditions for each of the following equations of the model and decide about identification

$$Y_1 = 3Y_2 - 2X_1 + X_2 + U_1 \rightarrow (1)$$

$$Y_2 = Y_3 + X_3 + U_2 \qquad \rightarrow (2)$$

$$Y_3 = Y_1 - Y_2 - 2X_3 + U_3 \rightarrow (3)$$

Oder condition for equation (1)

Here G=Number of equations in the model=3

K=Number of variables in the model=6

M=Number of variables in the equation=4

$$K - M = 6 - 4 = 2$$

$$G - 1 = 3 - 1 = 2$$

$$K - M = G - 1$$

Hence equation (1) is just identified.

Oder condition for equation (2)

Here G=3, K=6, M=3

Hence $K - M > G - 1$ equation (2) is over identified.

Oder condition for equation (3)

Here G=3, K=6, M=4

Hence $K - M = G - 1$ equation (3) is just identified.

Now we check the rank correlation for each of the above equations in a systematic way as shown below.

First write all the equations as under

$$Y_1 - 3Y_2 - 2X_1 - X_2 - U_1 = 0 \rightarrow (1)$$

$$Y_2 - Y_3 - X_3 - U_2 = 0 \qquad \rightarrow (2)$$

$$Y_3 - Y_1 + Y_2 + 2X_3 - U_3 = 0 \rightarrow (3)$$

Rank condition for equation (1)

Equation	Y_1	Y_2	Y_3	X_1	X_2	X_3	U_1	U_2	U_3
✔ (1)	1	-3	0	2	-1	0	-1	0	0
(2)	0	1	-1	0	0	-1	0	-1	0
(3)	-1	1	1	0	0	2	0	0	-1

(Delete the row for equation (1) and columns for the variables in the equations (1))

We get matrix $A = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$

$$|A| = -2 + 1 = -1 \neq 0$$

$$\therefore \rho(A) = 2$$

$G = 3$ $\quad\therefore \rho(A) = G - 1 \Rightarrow$ Equation (1) is Identified.

(Combining order and rank condition, we can decide that equation (1) is identified).

Rank condition for equation (2)

Equation	Y_1	Y_2	Y_3	X_1	X_2	X_3	U_1	U_2	U_3
(1)	1	-3	0	2	-1	0	-1	0	0
✓(2)	0	1	-1	0	0	-1	0	-1	0
(3)	-1	1	1	0	0	2	0	0	-1

Result matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 0 \end{pmatrix} : 2 \times 3$

Find $\rho(A)$.

$\Delta_1 = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2$, $\Delta_2 = \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = 0$, $\Delta_3 = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1$

At least one minor of order $G - 1 = 2$ is non-zero.

$\therefore \rho(A) = 2 = G - 1$

$\therefore$ Equation (2) is identified.

(From order and rank condition, equation (2) is identified)

Rank condition for equation (3))

Equation	Y_1	Y_2	Y_3	X_1	X_2	X_3	U_1	U_2	U_3
(1)	1	-3	0	2	-1	0	-1	0	0
(2)	0	1	-1	0	0	-1	0	-1	0
✓ (3)	1	1	1	0	0	2	0	0	1

Resulting matrix $A = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$

$|A| = 0$, $\rho(A) = 0$

$G - 1 = 2$

$\rho(A) < G - 1$

∴ Equation (3) is identified.

(From both the conditions, we find that equation (3) is unidentified)

Example: 2 Apply order and rank conditions for identifying the following model.

$D = a_0 + a_1P_1 + a_2P_2 + a_3\gamma + a_4t + U \;\rightarrow (1)$

$s = b_0 + b_1P_1 + b_2P_2 + b_3C + b_4t + w \;\rightarrow (2)$

$D = s \qquad\qquad\qquad\qquad\qquad\qquad \rightarrow (3)$

Write term as

$D - a_0 + a_1P_1 + a_2P_2 + a_3\gamma + a_4t + U = 0 \rightarrow (1)$

$s - b_0 + b_1P_1 + b_2P_2 + b_3C + b_4t + w = 0 \;\rightarrow (2)$

$D - s = 0 \qquad\qquad\qquad\qquad\qquad\qquad \rightarrow (3)$

Order condition for equation (1)

$G = 3, K = 7, M = 5, K - M = 7 - 5 = 2, G - 1 = 3 - 1 = 2$
$K - M = G - 1 \Rightarrow Equation\ (1)\ is\ just\ Identified$

Order condition for equation (2)

$G = 3, K = 7, M = 5, K - M = 7 - 5 = 2, G - 1 = 3 - 1 = 2$

$K - M = G - 1 \Rightarrow Equation\ (2)\ is\ just\ Identified$

Rank condition for equation (1)

Equation	D	S	P_1	P_2	Y	C	t	U	V
✔ (1)	1	0	$-a_1$	$-a_2$	$-a_3$	0	a_4	-1	0
(2)	0	1	$-b_1$	$-b_2$	0	$-b_3$	$-b_4$	0	-1
(3)	1	-1	0	0	0	0	0	0	0

$A = \begin{pmatrix} 1 & -b_3 \\ -1 & 0 \end{pmatrix}$ $|A| = -b_3 \neq 0$ (Assume $b_3 \neq 0$)

$Hence\ \rho(A) = 2$

$\rho(A) = G - 1 = 2$ Hence equation (1) is identified.

(Both conditions decide that equation 1 is identified)

Rank condition for equation (2)

Equation	D	S	P_1	P_2	Y	C	t	U	V
(1)	1	0		$-a_2$	$-a_3$	0	a_4	-1	0
~~(2)~~	~~0~~	~~1~~	$\cancel{-b_1}$	$\cancel{-b_2}$	~~0~~	$\cancel{-b_3}$	$\cancel{-b_4}$	~~0~~	~~1~~
(3)	1	-1	0	0	0	0	0	0	0

$A = \begin{pmatrix} 1 & -a_3 \\ -1 & 0 \end{pmatrix}$ $|A| = -a_3 \neq 0$ (Assume $a_3 \neq 0$)

$Hence\ \rho(A) = 2$

$\rho(A) = G - 1 = 2$ Hence equation (2) is identified.

(Both conditions refers to the decision of identification)

Estimation Methods for simultaneous Equations System Model

We shall discuss here briefly two methods

1) Indirect Least Squares Method
2) Two stage Least Squares Method

Indirect Least Squares Method (ILS Method)

This method of estimation is useful to estimate the parameters of just (or exact) identified equation. The estimators so obtained are called **Indirect Least Squares Estimators (ILSE).**

1. First obtain the reduced form equations. They are obtained from the structural equations in such a manner that the dependent variable in each equation is the only endogenous variable and is a function of the predetermined variates the stochastic error terms.
2. Apply OLS method to the reduced form equations individually. The estimates thus obtained are consistent.
3. Obtain the estimates of the original structural coefficients from the estimated reduced form coefficients. If the equation is exactly identified, there is one to one relation between structural coefficients and reduced from coefficients. Hence unique solution can be obtained.

Properties of ILSE

The estimates of the reduced form coefficients are consistent and under proper assumptions they are unbiased or asymptotically efficient.

ILSE has all those properties of estimates of reduced form coefficients for large samples. In small samples generally these estimators are biased. For large samples they become consistent also. We show this method by one illustration.

Illustration

Consider the model

Demand : $Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 X_t + U_{1t} \quad \rightarrow (1)$

Supply: $Q_t = \beta_0 + \beta_1 P_t + U_{2t} \qquad \rightarrow (2)$

Equilibrium: $D = S \qquad \rightarrow (3)$

Here Q=quantity, P= price, X= Income

It can be shown that demand function is not identified but supply function is just identified. We apply ILS method to estimate supply function. We obtain reduced form equations as under

$$P_t = \pi_0 + \pi_1 X_t + w_t \qquad \rightarrow (4)$$

$$and \quad Q_t = \pi_2 + \pi_3 X_t + \vartheta_t \qquad \rightarrow (5)$$

Applying OLS method we obtain the following estimates for π_0, π_1, π_2 and π_3 as given by

$\hat{\pi}_1 = \sum p_t x_t / \sum x_t^2$

Here $p_t = P_t - \bar{P}, \;\; q_t = Q_t - \bar{Q}, \; x_t = X_t - \bar{X}$

$\hat{\pi}_0 = \bar{P} - \hat{\pi}_1 \bar{X}$

$\hat{\pi}_3 = \sum q_t x_t / \sum x_t^2$

$\hat{\pi}_2 = \bar{Q} - \hat{\pi}_3 \bar{X}$

Hence we get $\hat{\beta}_0 = \hat{\pi}_2 - \hat{\beta}_1 \hat{\pi}_0$

$\hat{\beta}_1 = \hat{\pi}_3 / \hat{\pi}_1$

Which are ILSE of β_0 and β_1

Two Stage Least Squares Method

This method is useful for estimating over identified equation. We shall study this method by considering an illustration.

Illustration

Consider the following model

Income function

$$Y_{1t} = \beta_{10} + \beta_{11}Y_{2t} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + U_{1t} \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Money supply function

$$Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + U_{2t} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Where Y_{1t} =Income, Y_{2t} =Stock of money

X_{1t} =Investment expenditure

X_{2t} = Govt. expenditure on goods and services

(X_1 and X_2 are exogenous variables)

It can be seen that Income equation is unidentified whereas money supply function is over identified.

We can not use OLS method for equation (2) because due to over identification there are two estimates for β_{21} based upon reduced form equations.

We must use separate method for estimating money supply function, which is called **Two stage Least squares method (2 SLS method).** We indicate it in following steps.

Step 1 (First Stage)

We can get reduced form equation as under

$$Y_{1t} = \pi_0 + \pi_1 X_{1t} + \pi_2 X_{2t} + \vartheta_t$$

Where π_0, π_1 and π_2 are the reduced form coefficients.

To get rid of the likely correlation between Y_1 and U_2, regress first Y_1 on all the predetermined variables in the whole system, not just that equation. Here this means that regressing Y_1 on X_1 and X_2 gives

$Y_{1t} = \hat{\pi}_0 + \hat{\pi}_1 X_{1t} + \hat{\pi}_2 X_{2t} + \hat{U}_t$.................... (3)

where $\hat{U}_t$ are the usual OLS residuals

Hence we get

$\hat{Y}_{1t} = \hat{\pi}_0 + \hat{\pi}_2 X_{1t} + \hat{\pi}_2 X_{2t}$.............................. (4)

Where $\hat{Y}_{1t}$ is an estimate of the mean of Y_1 conditional upon the fixed X's.

Considering relations (3) and (4) we get

$Y_{1t} = \hat{Y}_{1t} + \hat{U}_t$.................................. (5)

Step 2 (Second Stage)

Since our aim is to deal with over identified money supply function given in (2) above, we get from (5) and (2), the following relation

$$Y_{2t} = \beta_{20} + \beta_{21}(\hat{Y}_{1t} + \hat{U}_t) + U_{2t}$$

$$= \beta_{20} + \beta_{21}\hat{Y}_{1t} + (U_{2t} + \beta_{21}\hat{U}_t)$$

$$= \beta_{20} + \beta_{21}\hat{Y}_{1t} + U_t^* \text{.......................} \quad (6)$$

Where $U_t^* = U_{2t} + \beta_{21}\hat{U}_t$

It can be shown that in large samples, $\hat{Y}_{1t}$ in equation (6) is uncorrelated with U_t^*. Hence we can apply OLS method to equation (6). Applying OLS regression to (6) we get $\hat{\beta}_{20}$ and $\hat{\beta}_{21}$ which are the 2SLS estimators of β_{20} and β_{21}.

This gives consistent estimators.

Some special feature of 2SLS method

1) It can be applied to individual equation in the system irrespective of any other equation in the system. Due to this the method is very common, popular and economical.
2) ILS method provides multiple estimates of the parameters, whereas 2SLS method provides unique estimates of the parameters.

3) It is easy to apply this method when we know about the exogenous or predetermined variables in the system.
4) It is a special method for over identified equations but it can also be used for just identified equations.
5) If R^2 in stage 1 is very high OLSE and 2 SLSE will be very close. If R^2 in stage 1 is very low, 2SLSE will be meaningless.

(**Note** In this representation of simultaneous equations system model we have not considered very rigorous mathematical results and expressions. Similarly we have also not referred to some other specialised tests and advances in the concerned subject.)