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Game Theory

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Lecture No. & Title:	Lecture – 1 (One) Game Theory

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1. Introduction

The competitive situations with two or more competitors, having conflicting interest and where the action of one depends upon the action taken by other, are known as **competitive games**. The competitors are called **players**. A player may be individual, a group of individuals or an organization.

A few examples of the competitive situations where theory of games may be used are: Two or more candidates contesting an election with the objective of winning with more votes; Marketing campaigns between competitive business organizations, etc.

If a game involves only two players, then it is called a **two-person game**. And if the numbers of players are more than two, the game is called **n-person game**.

If in a game the gains of one player are exactly equal to the losses of another player, so that the sum of gains and losses equal to zero, then the game is called a **zero sum game**.

2. Strategy

The strategy for a player is the list of all possible actions that he will take for every outcome that might arise. It is assumed that the rules managing the choices are known in advance to the players. Generally two types of strategies are employed by players in a game: Pure Strategy and Mixed strategy

(a)**Pure strategy**: It is a decision rule which is always used by the player to select the particular course of action. Thus each player knows in advance of all strategies out of which he always selects only one particular strategy irrespective of the strategy others may choose. The objective of the players is to maximize gains and minimize losses.

(b)**Mixed strategy**: When both the players are guessing as to which course of action is to be selected on a particular circumstance with some fixed probability, it is a mixed strategic. The objective of the player is to maximize expected gains or to minimize expected losses.

Two - person zero sum game

A game with only two players, say player A and player B is called a two-person zero sum game, if say player A's gain is equal to the loss of player B, so that total sum is zero.

The payoffs in terms of gains or losses, when players select their particular strategies, can be represented in the form of a matrix, called the **payoff matrix**.

Since the game is zero-sum, gain of one player is equal to the loss of other and vice-versa. Here one persons payoff table would contain the same amounts in the payoff table of other player but with changed sign. Thus, it is sufficient to construct payoff of one player only.

If player A has m courses of actions say $A_1, A_2 \ldots\ldots\ldots.. A_m$ and player B has n courses of actions say $B_1, B_2 \ldots\ldots\ldots.. B_n$. Let a_{ij} be the payoff which player A's gains from player B if the player A chooses strategy i and player B chooses strategy j.

Player B

$B_1 \quad B_2 \quad \ldots \quad B_n$

$$\text{Player A} \quad \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_m \end{array} \begin{bmatrix} a_{11} & a_{12} \ldots \ldots a_{1n} \\ a_{21} & a_{22} \ldots \ldots a_{2n} \\ & \vdots \\ & \vdots \\ a_{m1} & a_{m2} \ldots \ldots a_{mn} \end{bmatrix}$$

Assumptions of the Game

- Each player has available to him finite number of possible courses of action.
- Player A attempts to maximize gains and player B minimizes losses.
- The decisions of both players are made individually prior with no communication between them.
- The decision are made simultaneously and also announced simultaneously.

Pure Strategies: Games with saddle point

An optimal solution of two-person zero- sum game is obtained by making use of minimax – maximin principle, according to which player A selects the strategy which maximizes his minimum gain. In similar way, player B selects the strategy which minimizes his maximum losses. The value of game is the maximum guaranteed gain to the player A or the minimum possible loss to player B and it is denoted by v .

When maximin value = minimax value, the corresponding pure strategies are called optimal strategies and the game is said to have a saddle point and game is strictly determinable.

In general maximin value ≤ value of the game ≤ minimax value.

If the value of game is zero, the game is said to be **fair**.

Rules for determining a saddle point

- Select the minimum element of each row of the payoff matrix and write them under 'row minima' heading. Then select the maximum element among these elements and enclose it in a rectangle $\boxed{}$.
- Select the maximum element of each column of the payoff matrix and write them under 'column maxima' heading. Then select the minimum element among these elements and enclose it in a circle .
- If the value of the element enclosed in a rectangle and circle is same then it is the value of the game and game is called strictly determinable. Corresponding element in the payoff is called the saddle point.

Mixed Strategies: Games without saddle point

A game which is not strictly determinable i.e. without saddle point is solved by adopting mixed strategies. The optimal strategy mixture for each player is determined by assigning to each strategy its probabilities of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choice of strategy.

A mixed strategy game can be solved by different methods such as algebraic method, calculus method, matrix minima method, graphical method and linear programming method. We shall here restrict it to 2*2 games and algebraic method.

Algebraic Method

Le the payoff matrix of player A is given as

Player B

Player A $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then the following formulae are used to find out the value of game and optimal strategies.

The optimal strategies for player A is $\begin{bmatrix} \boldsymbol{A_1} & \boldsymbol{A_2} \\ p_1 & p_2 \end{bmatrix}$ and optimal strategies for player B is $\begin{bmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \\ q_1 & q_2 \end{bmatrix}$.

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \; ; \qquad q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

and $p_2 = 1 - p_1$; $q_2 = 1 - q_1$

Value of game is $v = \dfrac{a_{11} * a_{22} - a_{12} * a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

3. Dominance rules

The rules of dominance are used to reduce the size of the payoff matrix. These rules help in deleting certain rows and/or columns of the payoff matrix which is inferior to atleast one of the remaining rows and/or columns in terms of payoffs to both the players.

Certain dominance principles are stated as follows:

1. If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th}, the k^{th} row is dominated by the r^{th} row. So the kth row can be deleted from the payoff matrix.
2. If all the elements of a column, say k^{th}, are greater than or equal to the corresponding elements of any other column, say r^{th}, the k^{th} column is dominated by the r^{th} column. So k^{th} column can be deleted from the payoff matrix.

3. If strategy k dominates the convex combination of some other pure strategies then one of the pure strategies involved in the combination may be deleted. The dominance will be decided as rules 1 and 2 stated above.

4. Examples with Solution

Example 1: Solve the following game using the maximum-minimax principle. The payoff matrix is given below:

$$\begin{bmatrix} 10 & 30 & 10 \\ 0 & -40 & -30 \\ 10 & 50 & -10 \end{bmatrix}$$

Solution:

	B_1	B_2	B_3	Row Minima
A_1	10	30	10	10* Maximum
A_2	0	-40	-30	-40
A_3	10	50	-10	-10
Column Maxima	10* Minimum	50	10* Minimum	

From the above, it is clear that the value of maximin coincide with the value of minimax; therefore a saddle point exists and can be determined. Thus, the solution to the game is:

i) The optimum strategy for player A is A_1

ii) The optimum strategy for player B is B_1 or B_3

iii) The value of game for player A is 10 and for player B is -10.

Example 2: For a game with the following payoff matrix, determine the optimal strategies and the value of the game:

Player B

Player A $\begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix}$

Solution:

	B_1	B_2	Row Minima
A_1	6	-3	-3* Maximum
A_2	-4	0	-4
Column Maxima	6	0* Minimum	

From the above, it is clear that the value of maximin does not coincide with the value of minimax; therefore a saddle point cannot be determined using maximin and minimax principle.

Using algebraic method we will obtain the probability for each strategy. The optimal strategies for player A is $\begin{bmatrix} \boldsymbol{A_1} & \boldsymbol{A_2} \\ p_1 & p_2 \end{bmatrix}$ and optimal strategies for player B is $\begin{bmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \\ q_1 & q_2 \end{bmatrix}$.

$$p_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$= \frac{0-(-4)}{(6+0)-(-3-4)} = \frac{4}{13}$$

; $p_2 = 1 - p_1 = \frac{9}{13}$

Thus optimal strategies for player A is $\begin{bmatrix} \boldsymbol{A_1} & \boldsymbol{A_2} \\ \frac{4}{13} & \frac{9}{13} \end{bmatrix}$

Now,

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{0 - (-3)}{(6 + 0) - (-3 - 4)} = \frac{3}{13}$$

$;q_2 = 1 - q_1 = \frac{10}{13}$

Thus optimal strategies for player B is $\begin{bmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \\ \frac{3}{13} & \frac{10}{13} \end{bmatrix}$.

Value of game is $v = \frac{a_{11} * a_{22} - a_{12} * a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$$= \frac{6 * 0 - (-4) * (-3)}{(6 + 0) - (-4 - 3)} = \frac{-12}{13}$$

Example 3: Solve the following game if the payoff matrix is given by:

$$\begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix}$$

Solution:

	B_1	B_2	B_3	Row Minima
A_1	1	7	2	1
A_2	6	2	7	2* Maximum

A_3	5	1	6	1
Column Maxima	6* Minimum	7	7	

From the above, it is clear that the value of maximin does not coincide with the value of minimax; therefore a saddle point cannot be determined using maximin and minimax principle.

Using dominance property the game will be reduced to 2*2 matrix. Clearly, the third row is dominated by second row and third column is dominated by first column. Thus, the deletion of third row and third column yields the reduced payoff matrix.

Player B

Player A $\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}$

Using algebraic method the above game is solved.

The optimal strategies for player A is $\begin{bmatrix} A_1 & A_2 & A_3 \\ \frac{2}{5} & \frac{3}{5} & 0 \end{bmatrix}$ and optimal strategies for player B is $\begin{bmatrix} B_1 & B_2 & B_3 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ and value of game is 4.

5. Summary

- The competitive situations with two or more competitors, having conflicting interest and where the action of one depends upon the action taken by the other, are known as **competitive games**. The competitors are called **players**. A player may be individual, a group of individuals or an organization.

- If a game involves only two players, then it is called a **two-person game**. And if the numbers of players are more than two, the game is called **n-person game**.

- **Two - person zero sum game:** A game with only two players, say player A and player B is called a two-person zero sum game, if say player A’s gain is equal to the loss of player B, so that total sum is zero.

- **Pure Strategies: Games with saddle point**

  When maximin value = minimax value, the corresponding pure strategies are called optimal strategies and game is said to have a saddle point and game is strictly determinable.

- **Mixed Strategies: Games without saddle point**

  A game which is not strictly determinable i.e. without saddle point is solved by adopting mixed strategies.

- **Dominance rules**

  The rules of dominance are used to reduce the size of the payoff matrix.