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[Big-M Simplex Method]

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Big–M Simplex Method

Introduction:

When the constraints in linear programming problem (lpp)are≤ types, e.g. $2X_1 + 3X_2 \leq 17$ simplex method is used. But when the constraints in the problem are ≥ or = types, e.g. $5X_1 + 6X_2 \geq 25$, or $12X_1 + 20X_2 = 100$. To solve such problem many methods are available. One of them is the Big-M method. This method is also known as penalty method.

Steps in Big-M method:

i] Check whether the objective function of the given lpp is to be maximize or minimized. If it is to be minimized then convert it into a problem of maximization using the relation.

Max. Z = - Min (-Z)

e.g. Mini $Z = 24X_1 + 28X_2$

then we can write : - Max. $Z = -24X_1 - 28X_2$

ii] Check whether all b_i's are non-negative. If any of the b_i's negative then multiply the corresponding constraints by (-1) so as to get all b_i's non negative.

e.g. $X_1 - 3X_2 + 5X_3 \leq -7$

then we convert the above constraint into,

$-X_1 + 3X_2 - 5X_3 \geq 7$

iii] Convert all inequality of the constraints into equality by introducing slack and/or surplus variable in the constraints.

Put the cost coefficient of these variables equal to zero in the objective function.

In case of introducing surplus variables to get the initial basic feasible solution we need to introduce artificial variable and the cost coefficient of two variable in the objective function is –M, where M is very large positive number.

e.g. (i) $5X_1 + 4X_2 \leq 2000$

=> $5X_1 + 4X_2 + S_1 = 2000$, here S_1 is slack variable.

(ii) $X_1 + X_2 \geq 300$

=> $X_1 + X_2 - S_2 + A_1 = 300$,

Here S_2 is surplus variable and A_1 is artificial variable and in the objective function we put '0' coefficient for S_1 and S_2 and $-M$ for A_1.

(iii) If the objective function is maxi. $Z = 3X_1 + 7X_2$ then with respect to the above constraint we can write it as maxi. $Z=3X_1+7X_2++ 0S_1+ 0S_2 -MA_1$

(iv) If the objective function is, Mini $Z = 3X_1 + 7X_2$ then we convert it as Maxi. $Z = -3X_1 - 7X_2 + 0S_1+ 0S_2 -MA_1$

iv] Prepare initial (first) table of the simplex method as shown below.

		C_j:	C_1	……	C_j	……	C_n	0	0	……	-M
C_B	Y_B	X_B	Y_1	……	Y_j	……	Y_n	S_1	S_2	……	A_1
0	S_1	b_1	a_{11}	……	a_{1j}	……	a_{1n}	1	0	……	0
. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .
0	A_1	b_m	a_{m1}	……	a_{mj}	……	a_{11}	0	0	……	1
	$Z_j- C_j$	Z_j	Z_1-C_1	……	Z_j-C_j	……	Z_n-C_n	……	……	……	……

Note: in column Y_B we write S_1 ,S_2 ,…, A_1. Such that the column S_1,S_2,…, A_1. make identity matrix.

v] In the above table we compute $Z_j- C_j=(C_B.Y_j) - C_j$.

i.e. $Z_j - C_j$ for column Y_1 = $[(0)(a_{11}) + (0)(a_{12}) +…. + (0)(a_{m1})] - C_1$

Z_j for column $X_B = (0)(b_1) + (0)(b_2) + (0)(b_m)$

vi] (a) If all $Z_j - C_j \geq 0$ then the initial basic feasible solution (in the column X_B) is an optimal solution.

(b) if at least one $Zj - C_j< 0$ then proceed on the next step 7.

vii] If there are more then one negative values of $Zj - C_j$ then choose the most negative of them.

Suppose it be $Z_r - C_r$ for some j = r. then check,

(a) If all $Y_{ir} \leq 0$ (i.e. all the values in column Y_r), then this is an unbounded solution to the given problem.

(b) If atleast one $Y_{ir} \geq 0$, then corresponding vector Y_r enter into the basis Y_B .

viii] compute ratio $\frac{X_{Bi}}{Y_{ir}}$; for all $Y_{ir}> 0$, and choose the minimum of them. Let the minimum of these ratio occurs for k^{th} raw. i.e. $\frac{X_{Bk}}{Y_{kr}}$ is minimum. Then vector Y_k will leave the basis and in column Y_B, write Y_r in place of k^{th} value of Y_B . here the element Y_{kr} (element of k^{th} raw & r^{th} column in the first table) is called pivotal number and k^{th} raw is called pivotal raw , and r^{th} column is called pivotal column.

ix] Then we compute other entries of the columns of X_B, $Y_1, Y_2, \ldots, Y_j, \ldots$ in the second simplex table using the following formulas.

(a) New pivotal raw = $\dfrac{\textit{old pivotal raw}}{\textit{pivotal number}}$

(b) Other new raw = old raw – (pivotal column coefficient x new pivotal raw)

x] Go to step 5 and replace the above procedure until either the optimal solution is achieved or there is an indication of unbounded solution.

The decision regarding the optimum feasible solution is taken as follows,

(i) All $Z_j - C_j \geq 0$ and there is at least one artificial variable in the basis with non-zero value then lpp has no feasible solution. But if artificial variable is present in the basis with zero value then the solution is considered as optimal.

(ii) If all $Y_{ir} \leq 0$ in column Y_r, for which $Z_r - C_r$ is most negative then there is an unbounded solution to the problem.

Let us consider the following problem to understand the Big-M method.

Example : 1

Minimum $Z = 4X_1 + X_2$

Subject to : $3X_1 + X_2 = 3$

$4X_1 + 3X_2 \geq 6$

$X_1 + 2X_2 \leq 3$

$X_1 , X_2 \geq 0$

First of all we convert the constraints in to equations using necessary slack, surplus and artificial variables.

(i) In constraint $3X_1 + X_2 = 3$

We need to add artificial variables, but no use of slack or surplus variables.

and we write it as $3X_1 + X_2 + A_1 = 3$

From second constraints we write $4X_1 + 3X_2 - S_1 + A_2 = 6$

And from 3^{rd} constraints we write $X_1 + 2X_2 + S_2 = 3$

Now we convert our objective function from minimization to maximization as,

-Maxi Z = Mini(-Z)

$= -4X_1 - X_2$

And adding slack & surplus variables with 0 cost coefficient and artificial variable with –M cost coefficient. We write the objective function as,

-Maxi. Z = $-4X_1 - X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Now we prepare the first simplex table as follows,

Table:1

		C_j:	-4	-1	0	0	-M	-M	
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	A_1	A_2	Ratio $\left(\frac{X_{Bi}}{Y_{ir}}\right)$
0	S_2	3	1	2	0	1	0	0	$\frac{3}{1} = 3$
-M	A_1	3	3	1	0	0	1	0	$\frac{3}{3} = 1$
-M	A_2	6	4	3	-1	0	0	1	$\frac{6}{4} = 1.5$
	Z_j-C_j	-9M	-7M+4	-4M+1	M	0	0	0	

In Table 1 we arrange the coefficient of 3rd constraint in first raw, the coefficient of 1st constraint in second raw and the coefficient of 2rd constraint in third raw. Then we calculate the values of Z_j and Z_j-C_j for the column of X_B and then after for all the columns respectively. Then in last column we calculate the ratio $\frac{X_{Bi}}{Y_{ir}}$. Here the most negative value of Z_j-C_j is for the column Y_1 so, it called pivotal column and minimum ratio for the second raw having numeric values So, it is called the pivotal raw. And the cross element of these pivotal column and pivotal raw is called pivotal number , which is 3, denoted by red colour.

Table:2

		C_j:	-4	-1	0	0	-M	-M	
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	A_1	A_2	Ratio $\left(\frac{X_{Bi}}{Y_{ir}}\right)$
0	S_2	2	0	$\frac{5}{3}$	0	1	$\frac{1}{-3}$	0	$\frac{6}{5} = 1.2$
-4	Y_1	1	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	3
-M	A_2	2	0	$\frac{5}{3}$	-1	0	$\frac{-4}{3}$	1	$\frac{6}{5} = 1.2$
	Z_j-C_j	-2M-4	0	$\frac{-5M}{3}-\frac{1}{3}$	M	0	$\frac{7M}{3}-\frac{4}{3}$	0	

New pivotal raw (new second raw)= $\dfrac{old\ pivotal\ raw(second\ raw)}{pivotal\ number}$

$= \frac{1}{3}[\ 3,\ \ 3,\ \ 1,\ 0,\ \ 0,\ 1,\ \ 0\ ]$

$= \ [\ 1,\ \ \ 1\ ,\ \ \frac{1}{3},\ \ 0\ ,\ \ \ 0,\ \ \frac{1}{3},\ \ \ 0]$

And new first raw = old first raw – pivotal column coefficient (new pivotal raw)

$$\begin{array}{rccccccc} & 3 & 1 & 2 & 0 & 1 & 0 & 0 \\ (-1) & (1 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0) \\ \hline = & 2 & 0 & \frac{5}{3} & 0 & 1 & \frac{1}{3} & 0 \end{array}$$

similarly 2nd new raw,

$$\begin{array}{rccccccc} & (6 & 4 & 3 & -1 & 0 & 0 & 1) \\ (-4) & (1 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0) \\ \hline = & 2 & 0 & \frac{5}{3} & -1 & 0 & -\frac{4}{3} & 1 \end{array}$$

Here Z_j-C_j is most negative for Y_2 column & minimum ratio for 1st as well as 3rd raw, we choose randomly one, one of them let us consider 3rd raw as pivotal raw. i.e. A_2 will leave the basis and Y_2 will enter the basis.

And we prepare Table 3 by calculating new pivotal raw & new other rows in similar way.

Table : 3

		C_j:	-4	-1	0	0	-M	-M	
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	A_1	A_2	Ratio $(\frac{X_{Bi}}{Y_{ir}})$
0	S_2	0	0	0	1	1	1	-1	0
-4	Y_1	$\frac{3}{5}$	1	0	$\frac{1}{5}$	0	$\frac{7}{9}$	$-\frac{1}{5}$	3
-1	Y_2	$\frac{6}{5}$	0	1	$\frac{-3}{5}$	0	$\frac{-4}{5}$	$\frac{3}{5}$	—
	Z_j-C_j	$\frac{-18}{5}$	0	0	$\frac{-1}{5}$	0	$M-\frac{104}{45}$	$\frac{1}{5}+$ M	

New first raw = old first raw – pivotal column coefficient x (new pivotal raw)

2 0 5/3 0 1 -1/3 0

(-5/3) (6/5 0 1 -3/5 0 -4/5 3/5)

= 0 0 0 1 1 1 -1

Similarly, New 2nd raw,

1 1 1/3 0 0 1/3 0

(-1/3) (6/5 0 1 -3/5 0 -4/3 3/5)

= 3/5 1 0 1/5 0 7/9 -1/5

Table : 4

		C_j:	-4	-1	0	0	-M	-M
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	A_1	A_2
0	S_2	0	0	0	1	1	1	-1
-4	Y_1	$\frac{3}{5}$	1	0	0	$-\frac{1}{5}$	$\frac{26}{45}$	0
-1	Y_2	$\frac{6}{5}$	0	1	0	$-\frac{3}{5}$	$\frac{-7}{5}$	0
	Z_j-C_j	$\frac{-18}{5}$	0	0	0	$\frac{7}{5}$	$\frac{-104}{45}+\frac{63}{45}+M$	M

2nd new raw =

3/5 1 0 1/5 0 7/9 -1/5

(-1/5) (0 0 0 1 1 1 -1)

=3/5 1 0 0 -1/5 26/45 0

Similarly, 3rd new raw,

6/5 0 1 -3/5 0 -4/5 3/5

(-3/5) (0 0 0 1 1 1 -1)

=6/5 0 1 0 -3/5 -7/5 0

Hera all Z_j-C_j ≥ 0, so we reached at optimal solution. And no artificial variable present in the basis with non zero level.
The optimal feasible solution is,

$X_1 = \frac{3}{5}$, $X_2 = \frac{6}{5}$ and Minimize $Z = -\left(-\frac{18}{5}\right) = \frac{18}{5}$.

Now we consider another example in which the problem has no feasible solution.

Example 2

Maximize $Z = 3X_1 + 2X_2$

Subject to the constraints

$2X_1 + X_1 \leq 2,\ 3X_1 + 4X_2 \geq 12,\ X_1, X_2 \geq 0$

First of all we convert the constraints in to equations using necessary slack, surplus and artificial variables.

(i) In constraint $2X_1 + X_2 \leq 2$

We need to add slack variable.

and we write it as $2X_1 + X_2 + S_1 = 2$

(ii) Forthe second constraints $3X_1 + 4X_2 \geq 12$

We should subtract surplus variable S_2 and add artificial variable $A_{1,}$ hence it becomes $3X_1 + 4X_2 - S_2 + A_1 = 12$

Now , we set our objective function as

Maximize $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 - MA_1$

Hence we can prepare first simplex table as

Table 1

		C_j:	3	2	0	0	-M	
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	A_1	Ratio $(\frac{X_{Bi}}{Y_{ir}})$
0	S_1	2	2	1	1	0	0	$\frac{2}{1} = 2$
-M	A_1	12	3	4	0	-1	1	$\frac{12}{4} = 3$
	Z_j-C_j	-12M	-3M-3	-4M-2	0	M	0	

Here column of Y_2 has most negative value of Z_j-C_j, which is -4M-2, so this column is called pivotal column.

The minimum value of the ratio (which is 2) in the last column is for the first raw, so the first raw is called pivotal raw.

The intersection value of the pivotal column and pivotal raw is called pivotal number, which is 1.

Table 2

		C_j:	3	2	0	0	-M
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	A_1
2	Y_2	2	2	1	1	0	0
-M	A_1	4	-5	0	-4	-1	1
	Z_j-C_j	-4M+4	5M+1	0	4M	M	0

Here first we compute new pivotal raw as follow

New pivotal raw (new first raw)= $\dfrac{old\ pivotal\ raw(firstraw)}{pivotal\ number}$

$$= \frac{1}{1}[2, \quad 2, \quad 1, \quad 1, \quad 0, \quad 0, \ ]$$

$$= \quad [\ 2, \quad 2, \quad 1, \quad 1, \quad 0, \quad 0, \ ]$$

And new second raw = old second raw − pivotal column coefficient ×(new pivotal raw)

$$\begin{array}{rllllll} & (12 & 3 & 4 & 0 & -1 & 1) \\ (-4) & (\ 2 & 2 & 1 & 1 & 0 & 0) \\ \hline = & 4 & -5 & 0 & -4 & -1 & 1 \end{array}$$

Here all Z_j-$C_j \geq 0$ and artificial variable A_1 is in the basis with values 4(non-zero).

Therefore the LPP has no feasible solution.

Summary:

Simplex method is used only when the constraints are less than or equal to type, but when the constraints are greater than or equal or exactly equal to types then big-M method is used. For such types of constraints we should use surplus and artificial variables. In objective function the cost coefficient for surplus variable is taken as zero, but for artificial variable the cost coefficient is considered as very large number (say M) with negative sign.