[Academic Script]
[Linear Programming Problem]

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Linear programming problem

1. Introduction:

Many management decision involve regarding to make products of service involve the effective use of an organizations resources. The resources may include machinery, labour, money , time , space, raw materials etc. linear programming is a widely used mathematical modelling technique, which helps managers in planning & decision making relative to allocation resources. Here programming means modelling and solving a problem mathematically.

Linear programming problems are based on the main four steps.

i) Determine the objective function, the objective function is the function associated with maximization profit or minimization of cost.
ii) Determine the constrains or restrictions on the basis we can purse our objective.
iii) There must be alternative causes of action to choose from.
iv) The objective function and constrains must be expressed in terms of linear equations or inequalities.

Assumptions:

i) Numbers in the objective and constrains are known with certainty and do not change during the period of study.
ii) The proportionality exists in the objective and constrains. E.g. If production of units of a product required 4 units of raw material then making 5 units of that product uses 20 units of the raw material.
iii) Total of all activities equals the sum of the individual activities. e.g. If the objective function is max $Z = 2X_1 + 3X_2 + 5X_3$ and if one unit of each 3 types of products is produced then $Z = 2 + 3 + 5 = 10$.
iv) The solutions need not be in whole numbers(integers).
v) All numbers or its answers are non negative.

2. Formulation of LPP:

Formulation of lpp that means developing a mathematical model to represent the managerial problem in the following general form:

Max (min) $Z = C_1X_1 + C_2X_2 + C_3X_3$

Subject to the constraints,

$a_{11}X_1 + a_{12}X_2 + \ldots\ldots + a_{1n}X_n \le (\ge)\ b_1$

$a_{21}X_1 + a_{22}X_2 + \ldots\ldots + a_{2n}X_n \le (\ge)\ b_2$

. .

..

$a_{m1}X_1 + a_{m2}X_2 + \ldots\ldots + a_{mn}X_n \le (\ge)\ b_m$

$X_1, X_2, \ldots\ldots, X_n \ge 0$

Let us consider the following example to understand the formulation process of lpp.

Example : 1

A company produces two types of fans: small fans and large fans. Both require a certain amount of wiring and drilling. Each large fan takes 3 hours of wiring and 2 hours of drilling while each small fans must go through 2 hours of wiring and 1 hours of drilling. During the next production period, 240 hours of wiring time are available and upto 140 hours of drilling time may be used. Each assembled large fan sold yields a profit of 300 routers. Each assembled small fan may be sold for a profit of 200 rupees. Formulate the problem to find the best combination of large and small fans yields the maximum profit.

Solution: let us prepare a summary table which helps us to formulate the problem:

	Hours required to produce 1 unit		Profit per unit
	Wiring	Drilling	
Small fan	2	1	200
Large fan	3	2	300
Available hours	240	140	

Let X_1: number of small fans to be produced.

X_2: number of large fans to be produced.

Now our objective function becomes (for maximization of profit)

$$\text{Max } Z = 200X_1 + 300X_2$$

To develop constraints we have given two types of restrictions on usages of total hours for wiring and for drilling.

$$\text{For Wiring : } 2X_1 + 3X_2 \leq 240$$

$$\text{For Drilling : } X_1 + 2X_2 \leq 140$$

To obtain meaningful solutions, the values for X_1 and X_2 must be non-negative numbers.

i.e. $X_1 \geq 0$, $X_2 \geq 0$.

Hence the whole problem may now be restated mathematically as,

Maximize profit (Z) : $200X_1 + 300X_2$

Sub. To: $2X_1 + 3X_2 \leq 240$..(1)

$X_1 + 2X_2 \leq 140$..(2)

$X_1 , X_2 \geq 0$..(3)

Note: i) The set values of X_1 and X_2 satisfying the restrictions (1) & (2) is called a solution of the problem.

ii) A solution of the problem which satisfy the non negativity condition (3) is called feasible solution of a problem.

iii) A feasible solution which maximize (or minimize) the objective function of the problem is called optimum(feasible) solution of the problem.

3. Methods for solving LPP:

Mainly two methods are used to solve the problem of linear programming.

(i) Graphical method. (ii) simplex method. (iii) Big-M simplex method

(i) Graphical Method:

Graphically method is used for the problem having two variables only, while the simplex method is used for the problem having 2 or more variables.

Let us see the graphical method to solve the above lpp.

To represent the first constraint $2X_1 + 3X_2 \leq 240$. Graphically we convert the inequality into an equality. i.e. in an equation as, $2X_1 + 3X_2 = 240$.

To plot this line on graph we use two points of this line viz (0,80) and (120,0) by taking first $X_1=0$ and second times $X_2=0$. Then plot the line on a graph as below.

Now to identify the region satisfying the inequality,

$$2X_1 + 3X_2 \leq 240$$

We take randomly one point say (0,0) and check it for inequality. i.e $0 + 0 \leq 240$.
Which satisfy the inequality , That means the region covering (0,0) is the region for given inequality.
Thus we get the region as given in the following figure.

Similarly, we find the region for the second constrain. $X_1 + 2X_2 \leq 140$ and get the region foe both the restriction as below.

And from the non-negativity condition our common region satisfying both the restrictions is the region OPQR. To find optimum solution we use verifies of the region O (0,0) , P(120,0) , Q(60,40) ,
R(0,70).
Putting the value of X_1 and X_2 in objective function for each vertices we get Z at each point
$Z_0 = 0$, $Z_p = 24000$, $Z_Q = 12000 + 12000 = 24000$. $Z_R = 21000$.
Here the maximum value of Z is 24000, which is positive for two vertices Q and R.
That means we have two optimum solution to the problem.

Frist : produce small fans = 60 , large fans = 40 , maxi.profit (Rs.) = 24000.
Second :product only small fans = 120 , maxi.profit (Rs.) =24000.

Note:

1) Some time we may not get common region satisfying all the restriction. In this case we say that no solution exist.
2) Some time we may have unbounded common region and it is not possible to determine the value of the objective function. In this case we say that there is unbounded solution to the problem.

(ii) Simplex Method :

Simple method is an iterative procedure to obtain optimum values of the objective function of any lpp in which each iteration brings a higher value for the objective function so that we are always near to the optimal solution in each iteration.

Let us define slack and surplus variable used in the simplex method.

To convert each inequality in to an equation such variables are used, e.g. consider the inequality, $2X_1 + 3X_2 < 15$

Here this is smaller then RHS so to make both the sides equal we have to add some quantity (S_1)to the lhs.

i.e. we write, $2X_1 + 3X_2 + S_1 = 15$

Here S_1 is slack variable. i.e. a variable by adding to the lesser part of the inequality, the inequality becomes equality is called a slack variable. Slack variable represent unused resources may be in 2.

Steps for simplex method :

i]Check whether the objective function of the given lpp is to be maximize or minimized. If it is tobe minimized then convert it into a problem of maximization using the relation.

Max. Z = - Min (-Z)

e.g. Mini $Z = 24X_1 + 28X_2$

then we can write : - Max. $Z = -24X_1 - 28X_2$

ii] check whether all b_i's are non-negative. If any of the b_i's negative then multiply the corresponding constraints by (-1) so as to get all b_i's non negative.

e.g. $X_1 - 3X_2 + 5X_3 \leq -7$

then we convert the above constraints into,

$-X_1 + 3X_2 - 5X_3 \geq 7$

iii] convert all inequality of the constraints into equality by introducing slack and or surplus variable in the constraints. Put the cost coefficient of these variables equal to zero in the objective function.

iv] Prepare initial(first) table of the simplex method as shown below.

		C_j :	C_1	C_2	……	C_j	……	C_n	0	0	……	0
C_B	Y_B	X_B	Y_1	Y_2	……	Y_j	……	Y_n	S_1	S_2	……	S_m
0	S_1	b_1	a_{11}	a_{12}	……	a_{1j}	……	a_{1n}	1	0	……	0
0	S_2	b_2	a_{21}	a_{22}	……	a_{2j}	……	a_{2n}	0	1	……	0
. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .
0	S_m	b_m	a_{m1}	a_{m2}	……	a_{mj}	……	a_{11}	0	0	……	1
	Z_j-C_j	Z_j	Z_1-C_1	Z_2-C_2	……	Z_j-C_j	……	Z_n-C_n	……	……	……	……

Note: in column Y_B we write S_1 ,S_2 ,…. Such that the column S_1,S_2,…. make identity matrix.

v]In the above table we compute Z_j =(C_B.Y_j) – C_j.
i.e. Z_j – C_j for column Y_1 = [(0)(a_{11}) + (0)(a_{12}) +…. + (0)(a_{m1})] – C_1
$Z_j$for column X_B = (0)(b_1) + (0)(b_2) + (0)(b_m)
vi] (a) If all Z_j – $C_j \geq 0$ then the initial basic feasible solution (in the column X_B) is an optimal solution.
(b) if atleast one Zj – $C_j < 0$ then proceed on the next step 7.
vii] If there are more then one negative values of Zj – C_j then choose the most negative of them.
Suppose it be Z_r – $C_r$for some j = r. then check,

(a) If all $Y_{ir} \leq 0$ (i.e. all the values in column Y_r), then this is an unbounded solution to the given problem.

(b) If atleast one $Y_{ir} \geq 0$, then corresponding vector Y_r enter into the basis Y_B .

viii] compute ratio $\frac{X_{Bi}}{Y_{ir}}$; for all Y_{ir}> 0, and choose the minimum of them. Let the minimum of these ratio occurs for k^{th} raw. i.e. $\frac{x_{Bk}}{Y_{kr}}$ is minimum. Then vector Y_k will leave the basis and in column Y_B, write Y_r in place of k^{th} value of Y_B . here the element Y_{kr} (element of k^{th} raw & r^{th} column in the first table) is called pivotal number and k^{th} raw is called pivotal raw , and r^{th} column is called pivotal column.

ix]Then we compute other entries of the columns of X_B, $Y_1, Y_2, \dots, Y_j, \dots$ in the second simplex table using the following formulas.

(a) New pivotal raw = $\frac{\overline{old\ pivotal\ raw}}{pivotal\ number}$

(b) Other new raw = old raw – (pivotal column coefficient * new pivotal raw)

x] Go to step 5 and replace the above procedure until either the optimal solution is achieved or there is an indication of unbounded solution.

Let us consider the following example to understand the above process.

Here we again consider our Example 1.

Maxi. $Z = 200X_1 + 300X_2$

Subject to. $2X_1 + 3X_2 \le 240$

$X_1 + 2X_2 \le 140$

$X_1, X_2 \ge 0$

(i) Check that objective function is maximization and all b_i's are positive. So we convert inequalities to equally us.

$2X_1 + 3X_2 + S_1 = 240$ ....(1)

$X_1 + 2X_2 + S_2 = 140$ ....(2)

Our objective function including slack variables S_1 and S_2 can be written as,

$$Z = 200X_1 + 300X_2 + 0S_1 + 0S_2$$

(ii) Now we prepare initial(first) table of the simplex method as follow:

		C_j :	200	300	0	0	
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	Ratio $(\frac{X_{Bi}}{Y_{ir}})$
0	S_1	240	2	3	1	0	240/3 = 80
0	S_2	140	1	2	0	1	140/2 = 70
	Z_j- C_j	Z_j=0	Z_1-C_1 = -200	Z_2-C_2 = -300	Z_3-C_3 = 0	Z_4-C_4 = 0	

Note that here last two column S_1 and S_2 make identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so we write S_1 and S_2 respectively in the column Y_B. in the column C_B we write cost coefficient of S_1 and S_2 from the new objective function , which are 0, 0 respectively.

Now we calculate $Z_j = C_BY_j$

For the column X_B, $Z_j = (0)(240) + (0)(140) = 0$

For the column Y_1, $Z_1\text{-}C_1 = C_BY_1 - C_1$
$= (0)(2) + (0)(1) -200$
$= -200$

For the column Y_2, $Z_2\text{-}C_2 = C_BY_2 – C_2$
$= (0)(3) + (0)(2) -300$
$= -300$

For the column S_1, $Z_3\text{-}C_3 = (0)(1) + (0)(0) -0$
$= 0$

For the column S_2, $Z_4\text{-}C_4 = (0)(0) + (0)(1) -0$
$= 0$

Thus we get last raw in the first simplex table.

Now as per the step 6, we check the values of Z_j-C_j.

Here we have two negative values for Z_j-C_j for j= 1,2.

The most negative value is -300 for Z_2-C_2.
∴r=2 i.e. second column is our pivotal column.

And the all the values of Y_{i2} are positive.
∴There is no unbounded solution.

Now according to step 8, we calculate the ratio $\frac{X_{Bi}}{Y_{i2}}$;$Y_{i2} > 0$, i=1,2.

The ratio is denoted in the last column of the first simplex table.
The minimum ratio is 70 for the 2nd raw.
∴The second raw is called pivotal raw.
And the element belongs to pivotal raw and pivotal column(Cross section value) = 2 is called pivotal number.
Now we prepare second simple table on replacing S_2 by Y_2 in the basis column Y_B . and write corresponding cost coefficient from the new objective function. The second table looks as follow.

		C_j :	200	300	0	0
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2
0	S_1					
300	Y_2					

To calculate the entries in both the rows we use step 9 as shown below.
First of all we find new pivotal raw.

New pivotal raw (new second raw) = $\frac{old\ pivotal\ raw}{pivotal\ number}$

= $\frac{1}{2}$[140, 1 , 2 , 0 , 1]
= [70 , ½ , 1 , 0 , ½]

To calculate the entries of other new raw we use the formula.
For the, new first raw = old first raw – pivotal column coefficient (new pivotal raw)
First raw and pivotal column, the cross element "3" is called pivotal column coefficient for the first raw.

	240	2	3	1	0
-3 (	70	½	1	0	½)
=	30	½	0	1	-3/2

Putting these results in the above second table, we get the second table as,

Table:2

		C_j :	200	300	0	0	
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2	Ratio=($\frac{X_{Bi}}{Y_{ir}}$)
0	S_1	30	½	0	1	-3/2	60
300	Y_2	70	½	1	0	½	140
	Z_j-C_j	21000	-50	0	0	150	

Now we calculate Z_j-C_j for second table as described above,

Z_j for X_B : 0(100) + 300(70) = 21000

$Z_1-C_1 = 0(1) + 300(1/2) - 200 = -50$

$Z_2-C_2 = 0(1) + 300(1) - 300 = 0$

$Z_3-C_3 = 0(1) + 300(0) - 0 = 0$

$Z_4-C_4 = 0(-1) + 300(1/2) - 0 = 150$

Here again we have one negative value of Z_j-C_j for j=1 i.e. $Z_1-C_1 = -50 < 0$So in this table our pivotal raw will be first raw. Now we compute the ratio $\frac{X_{Bi}}{Y_{ir}}$ which is minimum for first raw (min ratio =100). Here pivotal number becomes 1.

		C_j :	200	300	0	0
C_B	Y_B	X_B	Y_1	Y_2	S_1	S_2
200	Y_1	60	1	0	2	-3
300	Y_2	40	0	1	-1	2
	Z_j-C_j	24000	0	0	100	0

Here all Z_j-$C_j \geq 0$ so we reached at optimum feasible solution.

The solution can be obtained from the column Y_B& X_B as

Y_1=60 , Y_2=40 and Maxi. Z = 24000 (the values of Z_j for X_B).

More Than One Optimal Solution:

Looking at the final table if Z_j-C_j value is equal to 0 for a variable that is not in the solution mix, more than one optimal solution exists.

In the above example, we find that in the final table (3^{rd} table) that variable S_2 is not included in the basis but Z_j-C_j is 0 for column S_2 . which indicate that an alternate optimal solution exists for the problem.

SUMMARY :

Linear Programming problem hepls in planning & decision making relative to allocation resources.

Linear programming problems consists objective function, constraints or restrictions and non-negative conditions. Here the objective function and constraints all are linear. To solve the linear programming problem graphical method, simplex method are used. Graphical method is used for lpp having variable only.