Subject: Business Economics Course: B.A., 2nd Semester, Undergraduate.

Paper No: 202 Paper Title: Mathematics for Business Economics

Unit No.: 5 (Five) Title: Functions of two variables

Lecture No: 2 (Two) Title: Some More Aspects of Partial Differentiation

Academic Scripts

Introduction

After having understood the other P's of marketing mix, we will now understand the most important and the

In the previous module we learnt about the Level Curves, Monotonic Functions, Homogeneous Functions, Homothetic Functions and Partial Derivatives.

Now we take up some important results involving the Partial Derivatives; They are Young's theorem, Euler's theorem etc. In addition to this, you will also learn a method of finding Maxima / Minima of functions of several variable and Optimization of Function with a single equality constraint.

So let's begin.

Geometrical Interpretation of Partial Derivatives:

Consider the surface .

Picture REQUIRED

<< graphics displaying the geometry of partial derivative

Now, if we treat

as a constant, say

, then becomes which is a function of one independent variable

only.

Let's call .

dx Thus, the derivative of

function is actually the partial derivative w.r.t. variable of function.

surface is a curve.

Now the derivative i.e.

at any point on this curve represents the slope of the tangent to the curve .

Thus

represents the slope of the tangent to the curve, which is obtained by the intersection of plane and the surface

Similarly, represents the slope of the tangent to the curve, which is obtained by the intersection of plane and the surfac

$$\frac{}{\partial x \partial y}(x_0, y_0) = \frac{}{\partial y \partial x}(x_0, y_0).$$

Now let's discuss a few important results involving partial derivatives:

Young's Theorem:

Suppose is a real valued function of two independent variables and if has continuous second order partial derivatives at a point in its domain, then

i.e. The partial derivative of w.r.t. and the partial derivative of w.r.t. commute under composition operation of mapping.

Example: 1) Consider and verify the truth of Young's Theorem for it. Solution

i.e. $\left(\frac{\partial}{\partial x} \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial y} \frac{\partial}{\partial x}\right)$

i.e. $\frac{}{\partial x \partial y} = \frac{}{\partial y \partial x}$

∂x

∂y

So, the order of partial derivatives (i.e. $\frac{\partial}{\partial y \partial x} = \frac{}{\partial y}$ first with respect to and then with respect to

or vice versa) is immaterial here as per the $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}($ Young's theorem because the second order parti derivatives exist and

are continuous for . $\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2}{\partial y}$

Euler's Theorem:

Suppose is a homogeneous function of degree

and if

has continuous partial derivatives, then

$\frac{}{\partial x} \quad \frac{}{\partial y}$. This result is known as Euler's theorem;

Note that the Euler's theorem can be used to verify the truth of another similar resu which we state here as a corollary to Euler's theorem.

Corollary: Suppose is a homogeneous function of degree

and if

$\frac{}{\partial x^2} + 2xy \frac{}{\partial x \partial y} + y \frac{}{\partial y^2} = n(n-1)f$.

a

s

$f(x, y) = \frac{}{x+y}$. continuous partial

derivatives at least up to second order, then

Example: 2) Verify the Euler's theorem for Solution:

First of all, we will check whether the given function homogeneous or not.

$f(tx, ty) = \frac{}{tx+ty} = \frac{}{x+y} = t \; f(x, y)$

$\therefore f(x, y) = \frac{x^2 y^2}{x+y}$ is a homogeneous function of

$\therefore \frac{\partial f}{\partial x} = \frac{2xy^2(x+y) - x^2 y^2}{(x+y)^2} = \frac{xy^2(x+2y)}{(x+y)^2}$ and

Consider $x \frac{}{\partial x} +$ Hence the Euler's Theorem is verified for

Chain Rule:

Let , where

is a function of another variable

i.e. then the derivative of

with respect to

is equal to the derivative of

with respect to

$f(x, y) = \frac{}{x + y}$. , times the derivative of with respect to

written symbolically as

This rule is called **Chain Rule.** In general, For , if

$dx \quad dy \quad dx$.

is a function of another variable

i.e. then the

derivative of

$dX_n \quad dX_2 \quad dX_3 \quad dX_4 \quad \cdots \quad dX_n$

w.r.t.

is written symbolically as

This rule is called **Generalised Chain Rule.** Now, Let's solve an example to understand the usefulness of Chain Rule.

Example: 3) Differentiate with respect to

. Solution:

Let’s denote .

Now by Chain Rule,

Now let's discuss some definitions:

Explicitly Defined Function:

A function of the type where

is explicitly written in terms of

ms of

is called an Explicitly defined function.

$x^2 + 4$

Hence, dx

For Example:

Implicitly Defined Functions:

An equation of the form from which

cannot be written in the explicit form in terms of is called an Implicitly defined function.

For Example: $x^2 + 4y$. Here it can be noted that it is difficult to find the explicit expression for in terms of .

Convex Combination:

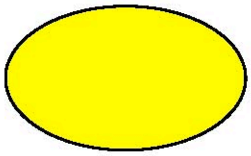
For any two points and of (or), its convex combination is given by for .

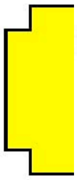
Geometrically, the set). represents a line segment joining the points and of (or

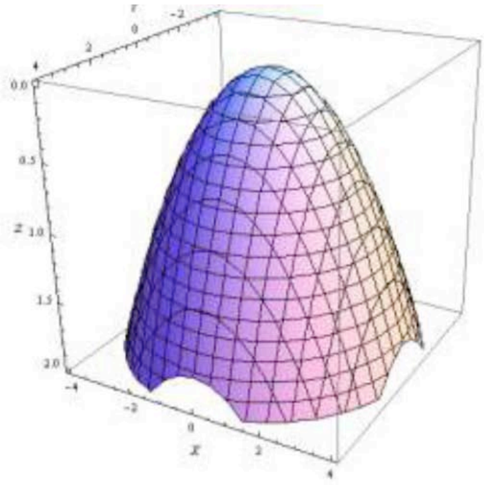
Convex Set:

A set S is said to be a Convex set if for every pair of points in set S, the line segment joining them completely lies in S. From this definition, it is clear that a line segment is an example of a Convex set. Also by convention we consider a singleton set as Convex set.

Example: 4) Identify Convex Sets from







the following list of shaded regions.

Solution:

Oval (figure-1) and Regular Octagon (figure-5) are both Convex Sets whereas figure-2, figure-3 and figure-4 are non-convex sets. Intuitively, the definition is clear, but a more rigorous mathematical definition can be provided. Here it follows.

A set S is said to be a Convex set if for every two points

in set S, belongs to S for all . Now, Let's discuss maxima and minima of functions of several variables:

Local Maxima and Local Minima of a Function of Two Variables:

For a function of one independent variable, an extreme value is represented graphically by the peak of a hill or the bottom of a valley in two-dimensional graph.

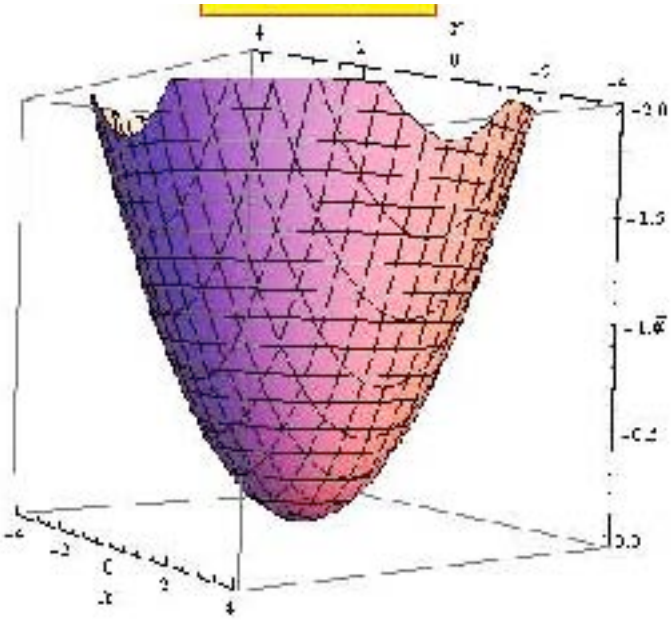
But the graph of a function of two independent variables become a surface in 3-space i.e.

Extreme values are still to be associated with the peaks and bottoms but the hills and valleys themselves now take on a three-dimensional character and they could be called as domes and bowls respectively.

n the module-6 of Unit-3, you earnt the use of derivatives in finding the extreme points of a function of single variable.

Similarly,

One can utilize the concept of partial derivatives to find points of maxima or minima of a function of two independent variables.



Necessary Condition for Maxima or Minima of a function of Two Independent Variables:

Necessary Condition for Maxima or Minima of a function

of Two Independent Variables and

is that the first order partial derivatives at these points must vanish.

.e.

∂x and But as specified, this is just the necessary condition for the existence of Extreme values. To learn about the

∂y

sufficient condition, let's fix up some notations:

Sufficient Condition for Maxima or Minima of a function of Two Independent Variables:

f both first order partial derivatives of a function

of two variables

and

are zero at some point

n the domain of

(i) is a point of Maxima? if , and

i.e. if and .

i.e. ∂x $- 24x + 2y - 6x = 0$ and ∂y

(ii) is a point of Minima? if ,

and

i.e. if and .

Saddle Point:

A point in the domain of a function is called a Saddle point if it is a stationary point but not a local extremum point.

i.e.

,

and

i.e. and

It should be noted that if for a stationary point

of a function

, then the second derivative test becomes inconclusive and the point

may be a point of local maxima or local minima or a saddle point.

∂y

Now let's solve an example;

Example: 5) Find extreme values of .

Solution:

∂x^2 $- 48x - 0$

First let's find all the first and second order partial derivatives of

$\partial x \partial y$

which are required.

∂y^2

and

Now we will find stationary points by solving the equations obtained by putting ∂x and ∂y .

and

$\left(3 \right.$.

Next, we will test these points for the optimality. For

:

is negative.

$$s = \left(\frac{}{\partial x \partial y} \right)_{(0,\ 0)} = 2$$

This is neither a point of maxima nor a point of minima. But it is a Saddle Point since .

For

$\left(3 \right.$:

is positive.

and

Therefore

$\left(3 \right.$ is a point of minima and

$$s = \left(\frac{}{\partial x \partial y} \right)_{\left(\frac{1}{3}, -\frac{1}{3}\right)} = 2$$

$$z\left(\frac{}{3}, -\frac{}{3}\right) = 8\left(\frac{}{3}\right) + 2\left(\frac{}{3}\right)\left(-\frac{}{3}\right) - 3\left(\frac{}{3}\right) + \left(-\frac{}{3}\right) + 1 = \frac{}{27}$$

Hence the local minimum value of

is

at the point $\left(3\right.$.

After having understood the method of finding local maxima or minima of a function of two variables, we will now learn, one more very important tool for finding optimal solution of a non-linear problem. Here it follows.

Optimization of Function with a Single equality constraint:

There are many practical situations where we have to maximise or minimise the given function of several variable subject to the given constraint.

i.e. a problem of type

Optimise subject to .

Here we apply the technique of **Lagrange's Undetermined Multipliers** which consist of the following steps.

Step-1: First we consider a function

The constant

in above equation is called Lagrange's Multiplier.

Step-2: Find first order partial derivatives of

i.e.

∂x ∂y.

Step-3: Solve

$g(x, y) = 0$, $\frac{}{\partial x} = 0$ **and** ∂y **simultaneously for**

and

. This will give the stationary

points of Lagrange's function

.

test.

Example: 6) Find extreme points of subject to . Solution:

The first step is to write the Lagrangian Function.

. For a stationary value of

$$U_y = 2y + 4\lambda \quad and$$

$$U_\lambda = x + 4y - 2$$

value is $(17, 17) = 289$.

, it is necessary that

i.e.

$\Rightarrow x \quad 2 \quad , y \quad 2x \quad , x + y = 2$ On solvir thes equatior simultaneous

we get $17, 17, 17$.

Thus the stationary point of function subject to is

$(17, 1$
and the stationary

Now by the Second Order Partial Derivative Test we will check whether this stationary point is a point of maxima or minima.

xx at every point in the domain of

$s = U_{xy} =$.

$t = U_{yy} =$

and .

Hence the stationary point

$(17, 1$ of the function subject to is the point of

minima and the required minimum value is

2. Now,

Let's summarise everything that we have done;

In this module, you have seen some famous results involving the partial differentiation like Young's theorem, Euler's theorem etc;

In addition to this you also learnt a method of obtaining maxima and minima of functions of several variables and finally you learnt an important and very useful application known as the method of Lagrange's Undetermined Multipliers.

With this I end up this module. Hope you liked and understood the topics covered here.