[Academic Script]

[Concept of Level set & Partial Differentiation]

Subject:	Business Economics
Course:	B.A., 2nd Semester, Undergraduate
Paper No. & Title:	Paper – 202 (Two Zero two) Mathematics for Business Economics
Unit No. & Title:	Unit - 5 Functions of two variables
Lecture No. & Title:	1(One): Concept of Level set & Partial Differentiation

In this module you will learn about the Level Curves, Level Surfaces, Monotonic Functions, Homogeneous Functions and Homothetic Functions. This will further follow the study of Partial Derivatives.

So let's start with the Level Curves.

Level Curves:

Let f be a function of two variables.

A collection of points defined by $L_c(f)=\{(x,y)\in R^2 : f(x,y)=c\}$ for some fixed $c\in R$ is called a **Level Set** of f for the value c.

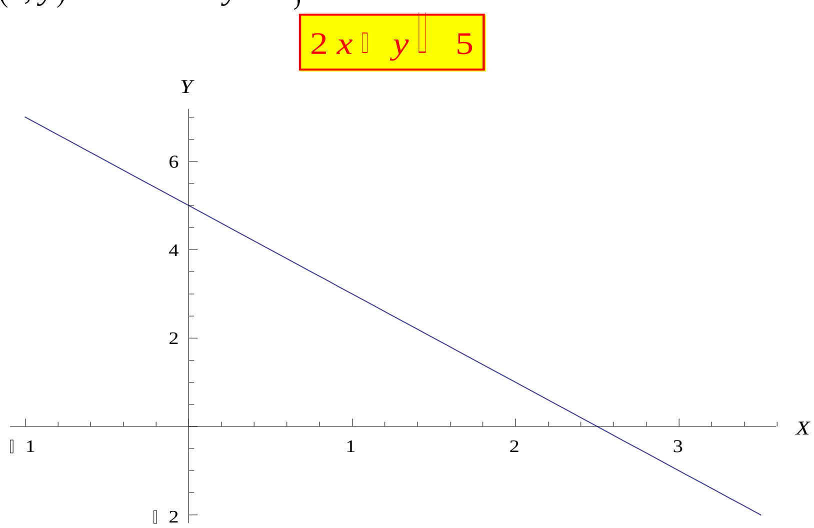
The graphical representation of a Level Set in R^2 is called a **Level Curve**.

Note that, if level curves of a function are drawn for various values of $c\in R$, then we can construct a topographic map of the function that gives us a good picture of the nature of the function.

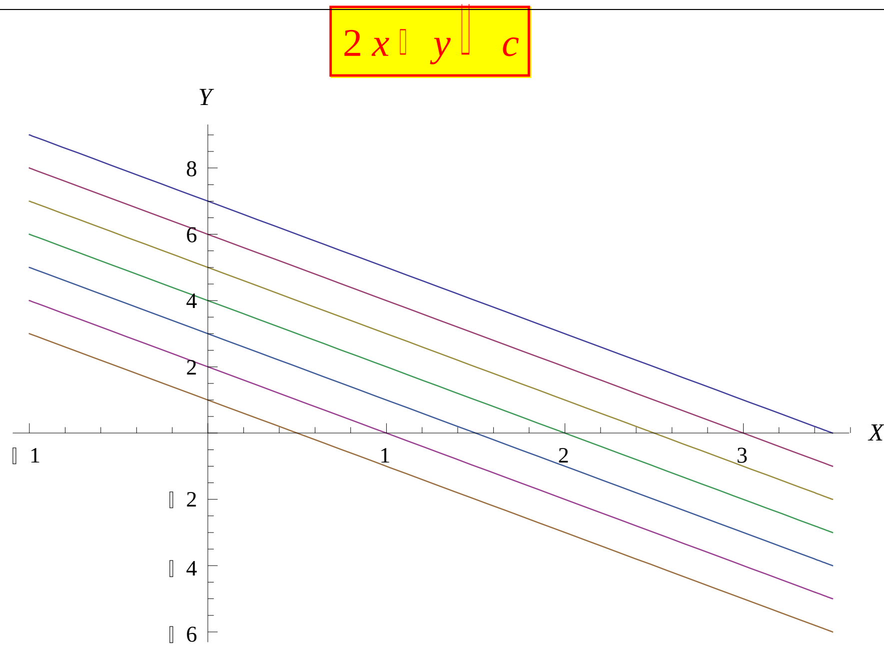
Example: 1) Construct a topographic map of $f(x,y)=2x+y$.

Sol: $L_5(f)=\{(x,y)\in R^2 : f(x,y)=5\}$

$=\{(x,y)\in R^2 : 2x+y=5\}$



This straight line is an example of a level curve of function $f(x,y)=2x+y$ for the value of c=5.



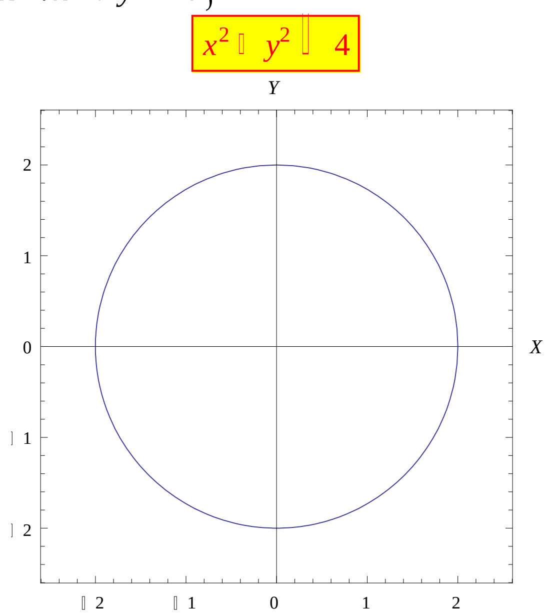
Here level curves of function $f(x, y) = 2x + y$ are drawn for values of $c = 1, 2, 3, 4, 5, 6$ *and* 7.

This topography of $f(x, y) = 2x + y$ represents a set of parallel lines having slope –2.

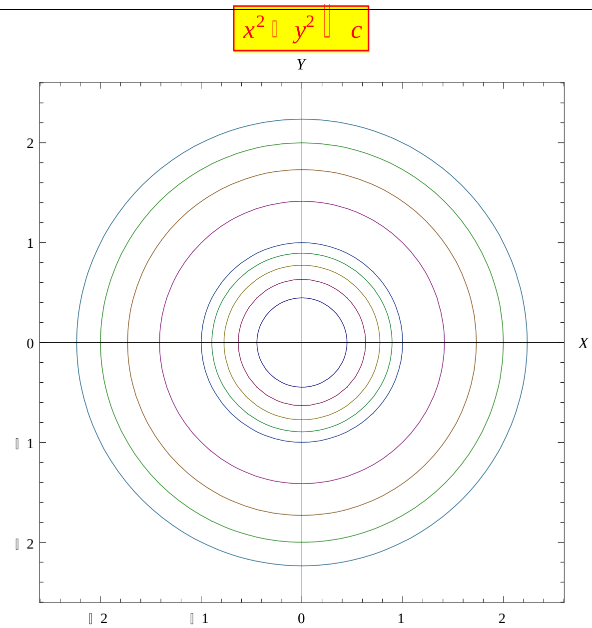
Example 2) Construct a topographic map of $f(x, y) = x^2 + y^2$.

Sol: $L_4(f) = \{(x, y) \in R^2 : f(x, y) = 4\}$

$$= \{(x, y) \in R^2 : x^2 + y^2 = 4\}$$



This circle is another example of a level curve of function $f(x, y) = x^2 + y^2$ for the value of c=4.



Here level curves of function $f(x,y)=x^2+y^2$ are drawn for values of $c=0.2,\ 0.4,\ 0.6,\ 0.8,\ 1,\ 2,\ 3,\ 4\ and\ 5$.

This topography of $f(x,y)=x^2+y^2$ describes a collection of concentric circles having centre at $(0,0)$.

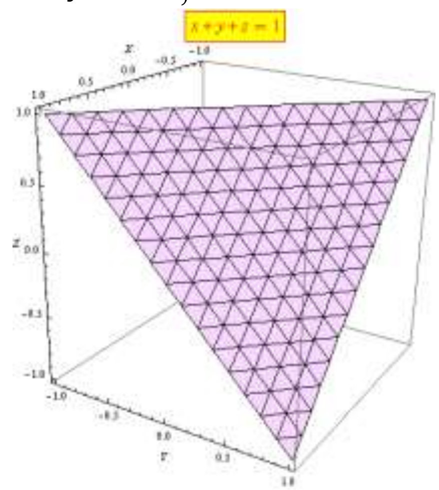
Similarly,

A collection of points defined by $L_c(f)=\{(x,y,z)\in R^3 : f(x,y,z)=c\}$ for some fixed $c\in R$ is called a Level set of f for the value c.

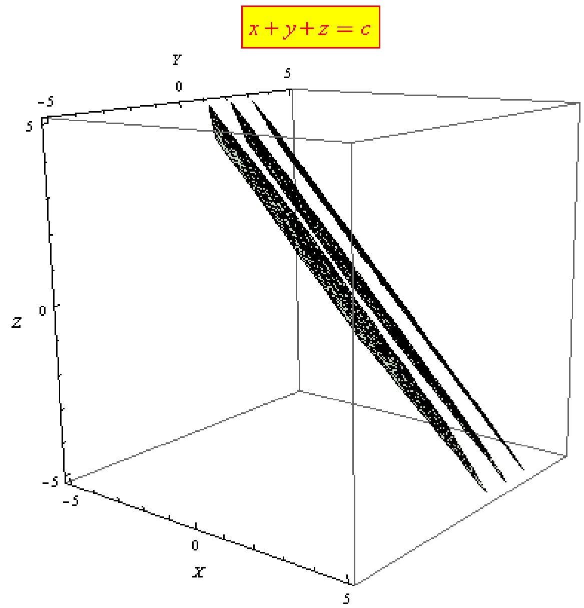
The graphical representation of a Level set in R^3 is called a **Level Surface**.

Example: 3) Construct a topographic map of $f(x,y,z)=x+y+z$.

Sol:
$$L_1(f)=\{(x,y,z)\in R^3 : f(x,y,z)=1\}$$
$$=\{(x,y,z)\in R^3 : x+y+z=1\}$$



This plane is an example of a level surface of function $f(x,y,z) = x+y+z$ for the value of c=1.

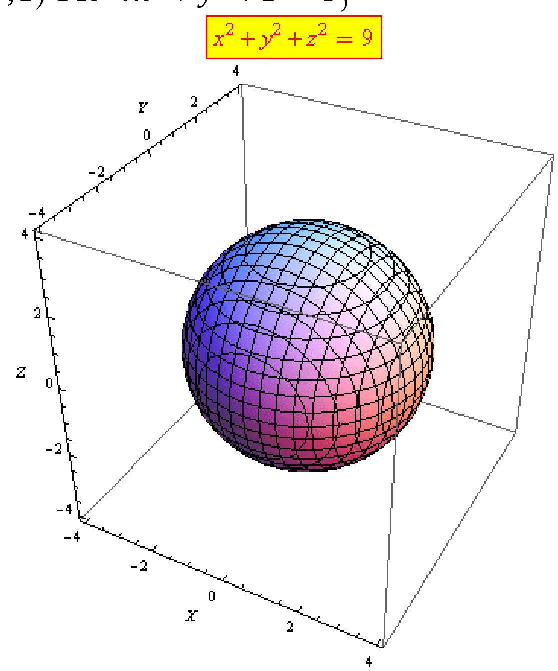


Here level surfaces of function $f(x,y,z) = x+y+z$ are drawn for values of $c = 1,\ 2,\ 3$.

This topography of $f(x,y,z) = x+y+z$ represents a collection of parallel planes.

Example: 4) Construct a topographic map of $f(x,y,z) = x^2+y^2+z^2$.

Sol: $$L_9(f) = \{(x,y,z) \in R^3 : f(x,y,z) = 9\}$$
$$= \{(x,y,z) \in R^3 : x^2+y^2+z^2 = 9\}$$



This sphere is an example of a level surface of function $f(x,y,z) = x^2+y^2+z^2$ for the value of c=9.

So, Can you now think of topography of $f(x,y,z)=x^2+y^2+z^2$?

Now let's discuss the concept of monotonic functions:

Monotonically Increasing Function:

A real valued function $f:R\to R$ is said to be monotonically increasing function if $x_1 \le x_2 \Rightarrow f(x_1)\le f(x_2)$ for all $x_1, x_2 \in R$.

Monotonically Decreasing Function:

A real valued function $f:R\to R$ is said to be monotonically decreasing function if $x_1 \le x_2 \Rightarrow f(x_1)\ge f(x_2)$ for all $x_1, x_2 \in R$.

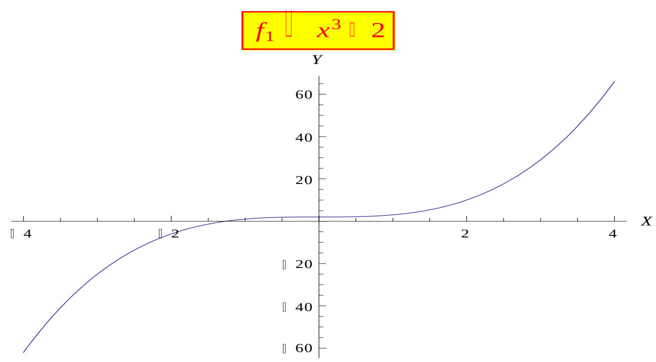
Example: 5) Show that $f_1(x)=x^3+2$ is a monotonically increasing function.

Sol: For any $x_1, x_2 \in R$, whenever $x_1 \le x_2$

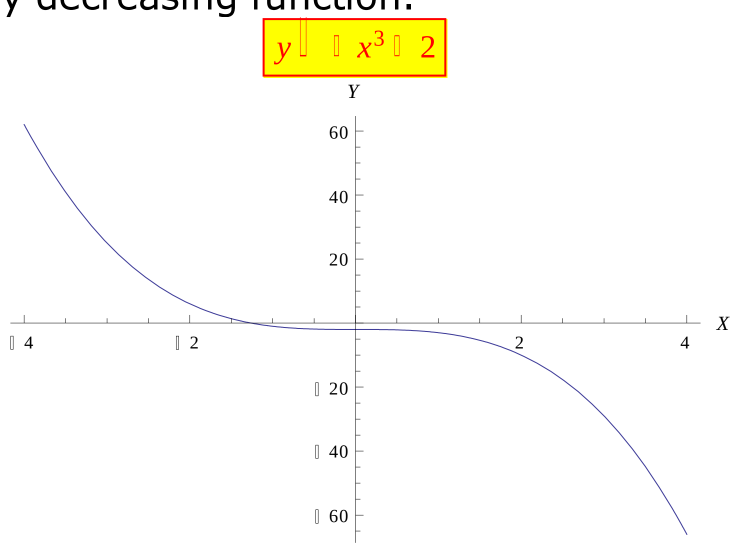
$\Rightarrow x_1^3 \le x_2^3 \quad \Rightarrow x_1^3+2 \le x_2^3+2$

$\Rightarrow f_1(x_1)\le f_1(x_2)$

Also, one can observe this behaviour from its graph.



Similarly, it can be observed that $y=-f_1(x)$ i.e. $y=-x^3-2$ is a monotonically decreasing function.



One important thing, to be noted here, is that it is practically not always possible to draw graph of functions to see their monotonic nature.
Fortunately, we are blessed with the concept of derivatives, which can also be utilised to detect whether a function is monotonic or not.

For this purpose,
Let's consider a well known result which states that a function of one independent variable is monotonic only if its first derivative does not change sign i.e. if the first derivative is positive throughout then the function is monotonically increasing function AND if the first derivative is negative throughout then the function is monotonically decreasing function.

This result is sometimes called the **First Derivative Test** to determine the Monotonic properties of a function.

Let's understand this with the help of an example.

Example: 6) Discuss the nature of function $f_2(x) = x^3 - 2x^2 + 4x - 1$.

Proof: Consider $y = f_2(x)$.

$$\therefore \; y = x^3 - 2x^2 + 4x - 1$$

$$\frac{dy}{dx} = 3x^2 - 4x + 4$$

$$= 3\left(x^2 - \frac{4}{3}x + \frac{4}{3}\right)$$

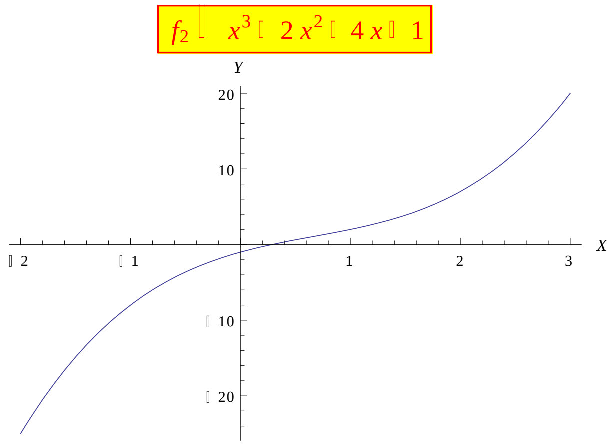
$$= 3\left[\left(x - \frac{2}{3}\right)^2 + \frac{8}{9}\right] \geq \frac{24}{9} = \frac{8}{3}$$

$$> 0$$

$\therefore \; \frac{dy}{dx} > 0$ for all values of x.

This implies that the function $f_2(x)$ is monotonically increasing function.

This increasing nature of $f_2(x)$ can also be seen from its graph.



Now, let's discuss homogeneous function and homothetic function.

Homogeneous Function:

A function $f : R^2 \to R$ satisfying the condition $f(tx, ty) = t^k f(x, y)$ for some fixed k is called a homogeneous function of degree k in two variables.

Example: 7) Show that $f_3(x, y) = x^4 + 2xy^3 + 4y^4$ is a homogeneous function of degree 4.

Proof:
$$\begin{aligned} f_3(tx, ty) &= (tx)^4 + 2(tx)(ty)^3 + 4(ty)^4 \\ &= t^4x^4 + 2t^4xy^3 + 4t^4y^4 \\ &= t^4\left(x^4 + 2xy^3 + 4y^4\right) \\ &= t^4 f_3(x, y) \end{aligned}$$
$$\therefore\ f_3(tx, ty) = t^4 f_3(x, y)$$

This implies that the function f_3 is homogeneous of degree 4.

Example: 8) Show that $f_4(x, y) = x^2 - 2xy - y^2$ is a homogeneous function of degree 2.

Proof:
$$\begin{aligned} f_4(tx, ty) &= (tx)^2 - 2(tx)(ty) - (ty)^2 \\ &= t^2x^2 - 2t^2xy - t^2y^2 \\ &= t^2\left(x^2 - 2xy - y^2\right) \\ &= t^2 f_4(x, y) \end{aligned}$$
$$\therefore\ f_4(tx, ty) = t^2 f_4(x, y)$$

This implies that the function f_4 is homogeneous of degree 2.

Homothetic Function:

A monotonic transformation of a homogeneous function is called homothetic function.

i.e. Let M be a monotonic function and H be a homogeneous function, then the function obtained by composition of M with H written as $M \circ H$ will be termed as a Homothetic Function.

Let’s illustrate this with an example.

Consider $H(x,y) = x^2 + 3xy$ and $M(x) = x^3 + 2$.
Here $H(x,y) = x^2 + 3xy$ is a homogeneous function of degree 2 and $M(x) = x^3 + 2$ is a monotonically increasing function.

$$\begin{aligned}\therefore g(x,y) &:= (M \circ H)(x,y) \\ &= M(H(x,y)) = M(x^2 + 3xy) \\ &= (x^2 + 3xy)^3 + 2\end{aligned}$$

$\therefore g(x,y) := (x^2 + 3xy)^3 + 2$ is a Homothetic function.
Now,
Let’s understand the concept of partial differentiation in brief.
Partial Derivatives:
Suppose f is a real valued function of two independent variables and (x_0, y_0) is a point in its domain.
The **First Order Partial Derivative of** f **with respect to** x at point (x_0, y_0) is defined as $\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$, provided this limit exists. This First Order Partial Derivative of f w.r.t. x at point (x_0, y_0) is denoted by $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$ or $D_x f$.
Similarly,
The **First Order Partial Derivative of** f **with respect to** y at point (x_0, y_0) is defined as $\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$, provided this limit exists. This First Order Partial Derivative of f w.r.t. y at point (x_0, y_0) is denoted as $f_y(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$ or $D_y f$.
This means that $D_x f$ shows a rate of change of f along X direction keeping the y value fixed and
similarly, $D_y f$ shows a rate of change of f along Y direction keeping the x value fixed.

The Second order Partial derivatives of a function of two independent variable f at point (x_0, y_0) are denoted by $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$, $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$, $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ and $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$.

Example: 9) If $f(x, y) = 2x^3 + xy$ then find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

Solution: Here $f(x, y) = 2x^3 + xy$

$\therefore \frac{\partial f}{\partial x} = 6x^2 + y$ ... differentiation of f keeping y-constant

$\frac{\partial f}{\partial y} = x$ ... differentiation of f keeping x-constant and

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x) = 1$ ... differentiation of $\frac{\partial f}{\partial y}$ keeping y-constant

Example: 10) If $f(x_1, x_2) = \frac{x_1^{\,2} + 1}{x_2^{\,2} + 2}$ then find the partial derivatives f_1' , f_2' and f_{12}''.

Solution: Here f_1' , f_2' , f_{12}'' means $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, $\frac{\partial^2 f}{\partial x_1 \partial x_2}$ respectively.

$$\therefore f_1' = \frac{\partial f}{\partial x_1} = \frac{2x_1}{x_2^{\,2} + 2}$$

$$f_2' = \frac{\partial f}{\partial x_2} = -\frac{2x_2\left(x_1^{\,2} + 1\right)}{\left(x_2^{\,2} + 2\right)^2}$$

$$\therefore f_{12}'' = \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right) = \frac{\partial}{\partial x_1}\left(-\frac{2x_2\left(x_1^{\,2} + 1\right)}{\left(x_2^{\,2} + 2\right)^2}\right)$$

$$= -\frac{2x_2}{\left(x_2^{\,2} + 2\right)^2} \cdot \frac{\partial}{\partial x_1}\left(x_1^{\,2} + 1\right) = \frac{-4x_1x_2}{\left(x_2^{\,2} + 2\right)^2}$$

Now,
Let’s summarise everything that we have done.
In this module, we discussed the concept of Level Curves, Level Surfaces, Monotonic Functions, Homogeneous Functions and Homothetic Functions.
Also we briefly discussed the partial derivatives and some examples on that. You will learn more on partial derivatives and its applications in the next module.

With this I end up this module.
Hope you liked and understood the topics covered here.