[Academic Script]

Manipulations of Matrices and Determinants

Subject:	Business Economics
Course:	B. A. (Hons.), 2nd Semester, Undergraduate
Paper No. & Title:	Paper – 202 Mathematics for Business Economics
Unit No. & Title:	Unit – 4 Linear Algebra
Lecture No. & Title:	Lecture – 2 Manipulations of Matrices and Determinants

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1. Some don't's about Matrix Algebra:

Note that we have defined addition and multiplication of Matrices. These operations are not behaving exactly as the way algebra or arithmetic of real and complex numbers go. So one has to be little more vigilant and should not be misguided by analogy while one is manipulating matrices.

Unlike in real or complex numbers, it is not true in general that $AB = BA$.

Example:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $and\ B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ then

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix}$

$while\ BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix}$

Clearly it follows that in general $AB \neq BA$

Unlike in real or complex numbers it may happen that $\boldsymbol{AB = 0}$**,** but neither $\boldsymbol{A = 0}$ **nor** $\boldsymbol{B = 0}$**.**

Example:

Let $\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{6} & \mathbf{4} \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{-4} \\ \mathbf{-3} & \mathbf{6} \end{bmatrix}$ then $\boldsymbol{A \neq 0}$ also $\boldsymbol{B \neq 0}$**.** But,

$\boldsymbol{AB} = \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{6} & \mathbf{4} \end{bmatrix}\begin{bmatrix} \mathbf{2} & \mathbf{-4} \\ \mathbf{-3} & \mathbf{6} \end{bmatrix} = \begin{bmatrix} \mathbf{6-6} & \mathbf{-12+12} \\ \mathbf{12-12} & \mathbf{-24+24} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

The law of cancellation is not valid here. That is one can't conclude $\boldsymbol{C = D}$ from $\boldsymbol{AC = AD}$**.**

Example:

Suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ then

$AC = \begin{bmatrix} 1+3 & 2+4 \\ 1+3 & 2+4 \end{bmatrix}$

$AD = \begin{bmatrix} 3+1 & 4+2 \\ 3+1 & 4+2 \end{bmatrix}$

Thus $AC = AD$, however $C \neq D$.

2. Determinant, Adjoint and Inverse of a Matrix:

Determinant of a square matrix is one of the very important concepts related to matrices. It is a single number derived from a given square matrix, which is useful at number of situations in Mathematics and its application. It can be defined in many ways. One of these definitions is "inductive definition". The phrase "inductive definition" is a technical word. But it roughly means that you are able to give appropriate definition for initial case. Then after, one is able to describe the procedure of moving to the next case from the current one. If this is done, one accepts that the definition is valid for all the positive integers or for all positive integers starting from the initial case.

Determinant of a 1×1 matrix $A = [a]$ is the number a itself.

Determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number $ad - bc$ i.e. the product of the entries on a main diagonal minus the product of entries on the other diagonal. The determinant of a matrix A is denoted by $|A|$. Thus we write,

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

The determinant of a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is defined in terms of determinant of 2×2 matrices. The definition is:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Therefore,

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$$

In the next example we shall soon see that there is a nice way to write down this expression in a way which can be easily applied and remembered.

In general, the determinant of a $n\times n$ square matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ is defined by:

$|\boldsymbol{A}| = \sum_{i=1}^{n} \boldsymbol{a_{ij}C_{ij}}$ **,** for any fixed j

where $\boldsymbol{C_{ij}}$ is the cofactor of $\boldsymbol{a_{ij}}$ defined by

$\boldsymbol{C_{ij}} = (-\mathbf{1})^{i+j}M_{ij}$

where M_{ij} is the minor of matrix $\boldsymbol{A}$ i.e. the determinant of the $(n-1)\times(n-\mathbf{1})$ matrix formed by suppressing i^th^ row and the j^th^ column of matrix $\boldsymbol{A}$**.** Note that it can be found by fixing any j, surprisingly all these values are same. The formula

$|\boldsymbol{A}| = \sum_{j=1}^{n} \boldsymbol{a_{ij}C_{\square j}}$ **,** for any fixed i, also can be used by fixing any i, again surprisingly all these values are same. The determinant of the Zero matrix is 0 and determinant of the Identity matrix is 1.

We shall find the determinant of a 3×3 matrix using both equivalent forms.

Example: We shall find $|\boldsymbol{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

First Form: We write down:

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{matrix}$$

Sum of the products of three diagonal entries with positive sign as suggested.

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{matrix}$$

Sum of the products of three diagonal entries with negative sign as suggested.

Determinant = $(1\times5\times9)+(2\times6\times7)+(3\times4\times8)$

- $(3\times5\times7)-(1\times6\times8)-(2\times4\times9)$

= 225-225

= 0

Second Form:

Here let us fix $i=2$, so that our formula for this case is:

$$|\boldsymbol{A}| = \sum_{j=1}^{3} \boldsymbol{a_{2j}C_{2j}}$$

$$= \boldsymbol{a_{21}C_{21}+a_{22}C_{22}+\ a_{23}C_{23}}$$

$$|\boldsymbol{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 4\left(-\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}\right) + 5\left(\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}\right) + 6\left(-\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}\right)$$

$$= \boldsymbol{4(6)+5(-12)+6(6)}$$

$$= \boldsymbol{0}$$

<u>Properties of Determinant:</u>

1. The value of the Determinant remains unaltered if rows and columns are transposed. That is $|\boldsymbol{A}| = |\boldsymbol{A^T}|$.
2. The value of the Determinant is multiplied by the constant λ if each element of the row or column is multiplied by $\boldsymbol{\lambda}$.
3. The determinant has a zero value if two rows or two columns are linearly dependent.
4. If elements of one row or column of a determinant are the sum of two numbers, then the natural law of addition holds. e.g.

$$\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \square_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

If one uses the properties number 3 and 4 one gets:

5. The value of the determinant remains unaltered if a multiple of one row or column is added to corresponding elements of some other row or column.
6. Determinant of the product of two matrices is the product of the determinant of these matrices. That is

   $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$

Adjoint of a Matrix:

Adjoint of a $n\times n$ square matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$, is defined and denoted by:

$Adj(A) = [C_{ij}]^T$

where $C_{ij} = (-1)^{i+j}M_{ij}$ and M_{ij} is the minor that is the determinant of $(n-1)\times(n-1)$ matrix obtained by suppressing the i^{th} row and j^{th} column of $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$

Example: Find $Adj\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right)$

From our definition it will be:

$$Adj\left(\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\right) = \begin{bmatrix} \begin{vmatrix}5 & 6\\8 & 9\end{vmatrix} & -\begin{vmatrix}4 & 6\\7 & 9\end{vmatrix} & \begin{vmatrix}4 & 5\\7 & 8\end{vmatrix} \\ -\begin{vmatrix}2 & 3\\8 & 9\end{vmatrix} & \begin{vmatrix}1 & 3\\7 & 9\end{vmatrix} & -\begin{vmatrix}1 & 2\\7 & 8\end{vmatrix} \\ \begin{vmatrix}2 & 3\\5 & 6\end{vmatrix} & -\begin{vmatrix}1 & 3\\4 & 6\end{vmatrix} & \begin{vmatrix}1 & 2\\4 & 5\end{vmatrix} \end{bmatrix}^T$$

Thus,

$$Adj\left(\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\right) = \begin{bmatrix} \begin{vmatrix}5 & 6\\8 & 9\end{vmatrix} & -\begin{vmatrix}2 & 3\\8 & 9\end{vmatrix} & \begin{vmatrix}2 & 3\\5 & 6\end{vmatrix} \\ -\begin{vmatrix}4 & 6\\7 & 9\end{vmatrix} & \begin{vmatrix}1 & 3\\7 & 9\end{vmatrix} & -\begin{vmatrix}1 & 3\\4 & 6\end{vmatrix} \\ \begin{vmatrix}4 & 5\\7 & 8\end{vmatrix} & -\begin{vmatrix}1 & 2\\7 & 8\end{vmatrix} & \begin{vmatrix}1 & 2\\4 & 5\end{vmatrix} \end{bmatrix} = \begin{bmatrix}-3 & 6 & -3\\6 & -12 & 6\\-3 & 6 & -3\end{bmatrix}$$

<u>Inverse of a Matrix:</u>

Inverse of a $n \times n$ square matrix $\boldsymbol{A}$ is another square matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$. It is not necessary that such a matrix $\boldsymbol{B}$ exists. If it exsits it is denoted by $\boldsymbol{A}^{-1}$, and in this case we say that matrix $\boldsymbol{A}$ is invertible. Thus when $\boldsymbol{A}$ is invertible we have

$\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$

It is clear from this that

$|\boldsymbol{AA}^{-1}| = |\boldsymbol{I}| = \boldsymbol{1}$

Therefore,

$|\boldsymbol{A}||\boldsymbol{A}^{-1}| = \boldsymbol{1}$

From which it follows that $|\boldsymbol{A}|$ and $|\boldsymbol{A}^{-1}|$ both are nonzero. In fact:

A is invertible if and only if the Determinant of $\boldsymbol{A}$ i.e. $|\boldsymbol{A}|$ is nonzero.

When $|\boldsymbol{A}| \neq \boldsymbol{0}$, the matrix A is also called nonsingular matrix. Thus, A is invertible iff $|\boldsymbol{A}| \neq \boldsymbol{0}$ iff A is nonsingular.

If A is nonsingular it is invertible and its inverse can be found by:

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

Example: If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Then

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

$$A^{-1} = \frac{1}{-2} Adj\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{-2}\left(\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T\right)$$

$$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

3. Rank of a Matrix:

Suppose $A = [a_{ij}]$, is a $m \times n$ matrix. We define the row-rank of a A as the maximum number of linearly independent row vectors of $= [a_{ij}]$. Similarly column-rank of A is the maximum number of linearly independent column-vectors of A. It can be proved that for any matrix A the row-rank and the column-rank of A are equal. This common number is called the rank of A and we shall denote it by rank(A). It is clear that: $rank(A) \leq min\{m, n\}$

We have seen that there is a link between determinant and linear independence. So we have another equivalent definition of a rank of a Matrix. The rank of a Matrix is the maximum order of a non-vanishing determinant that can be formed out of the given Matrix. As both these definitions are equivalent, rank found using any of the definition will be equal. So as per our convenience we can use any definition.

Example:

We shall find the rank of the 2×3 Matrix, using both definitions:

$$A = \begin{bmatrix} 2 & 3 & 8 \\ 4 & 6 & 16 \end{bmatrix}$$

First Method:

We first note that

$rank(A) \leq \min\{2,3\} = 2.$

Our second observation is that the two row-vectors $(2,3,8)\ and\ (4,6,16)$ are linearly dependent, as

$2(2,3,8) - 1(4,6,16) = 0$

Which is a non-trivial linear combination of $(2,3,8)\ and\ (4,6,16)$ that is equal to a zero vector. This means that we can't have two linearly independent row-vectors. On the other hand we do have one vector which is linearly independent. Thus row-rank of A is 1.

To find column-rank we start with the observation that the set of column-vectors $\{\ (2,4),\ (3,6),\ (8,16)\ \}$ is linearly dependent as:

$\mathbf{3(2,4) - 2(3,6) + 0(8,16) = 0}$ is a non-trivial linear combination which is zero.

Next we have to see if we can find two vectors out of these three which form a linearly independent set. But that also is not possible as:

$\mathbf{3(2,4) - 2(3,6) = 0}$

$\mathbf{8(3,6) - 3(8,16) = 0}$

$\mathbf{4(2,4) - 1(8,16) = 0}$

Thus none of the two column vectors give us a linearly independent set. Note that here each of the singleton set of the column vectors is linearly independent. Thus the maximum number of linearly independent column-vectors here is 1. Hence the column-rank also is one which was expected.

Second Method:

In the second method we just have to observe that all 2×2 sub-determinants here are 0. That is:

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 8 \\ 6 & 16 \end{vmatrix} = \begin{vmatrix} 2 & 8 \\ 4 & 16 \end{vmatrix} = 0$$

Thus the maximum order of non-vanishing sub-determinant here is one. Hence the rank of the given matrix A is 1.

4. Solutions to Simultaneous Linear Equations:

We started our discussion of Matrices and Vectors primarily with an objective of solving simultaneous linear equations.

Now we are in a position of carrying out the discussion.

We say that a system of equations is inconsistent if there is no common solution for the given system. If there is a common solution, unique or many, we say that the given system is consistent. Henceforth, by a solution of the system we shall always mean the common solution which satisfies all the equations under consideration.

Example:

$2x + 3y = 6$

$4x + 6y = 9$

This system is inconsistent as if there is a common solution (x, y) then the second equation can be written as $2(2x + 3y) = 9$

But because of the first equation we get absurd and untenable identity $2\times6 = 9$

Hence there is no solution for the system. Therefore the system is inconsistent.

Geometrically these equations represent two parallel lines in the xy-plane and hence there is no common solution.

Example

$2x + 3y = 6$

$x + y = 9$

This system of linear equations is consistent as, we can find a solution by substituting the value of y from the second equation into the first equation. One gets:

$2x + 3(9 - x) = 6$

$\therefore 2x + 27 - 3x = 6$

$\therefore -x = -21$

$\therefore x = 21$ and from the second equation one gets $y = -12$.

Check that $(x, y) = (21, -12)$ satisfies both the given equations. Hence the system is consistent.

Geometrically the above two equations represent two lines which are not parallel and therefore they must intersect. The point $(x, y) = (21, -12)$ is the unique point of intersection of these lines. Note that, here the solution is unique.

In general we have m equations in n variables, which can be written as:

$$[a_{ij}]\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & \dots & \boldsymbol{a_{1n}} \\ \boldsymbol{a_{21}} & a_{22} & \dots & \boldsymbol{a_{2n}} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

that is same as:

$\boldsymbol{Ax} = \boldsymbol{b}$ where $A = [a_{ij}]$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$.

Or if prefer we may also write these equations as:

$$\boldsymbol{x_1}\begin{bmatrix} \boldsymbol{a_{11}} \\ \boldsymbol{a_{21}} \\ \vdots \\ \vdots \\ \boldsymbol{a_{m1}} \end{bmatrix} + x_2\begin{bmatrix} \boldsymbol{a_{12}} \\ \boldsymbol{a_{22}} \\ \vdots \\ \vdots \\ \boldsymbol{a_{m2}} \end{bmatrix} + \cdots \boldsymbol{x_n}\begin{bmatrix} \boldsymbol{a_{1n}} \\ \boldsymbol{a_{2n}} \\ \vdots \\ \vdots \\ \boldsymbol{a_{mn}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

$$\boldsymbol{x_1 a_1} + \boldsymbol{x_2 a_2} + \cdots \boldsymbol{x_n a_n} = \boldsymbol{b}$$

$\boldsymbol{here\ a_i} = \begin{bmatrix} \boldsymbol{a_{1i}} \\ \boldsymbol{a_{2i}} \\ \vdots \\ \vdots \\ \boldsymbol{a_{mi}} \end{bmatrix}$ $\boldsymbol{for\ i = 1, 2 \ldots n}$ are the column vectors of the matrix.

We also wish to inform a nice result which means that:

A system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution iff the rank of $m \times n$ matrix A and the rank of augmented $m \times (n+1)$ matrix $[A, b]$ are equal. That is in other words, a system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution iff rank($\boldsymbol{A}$) = rank($\boldsymbol{A}, \boldsymbol{b}$).

When $m = n$, the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution if the square Matrix A is nonsingular. In this case inverse of A exists. And the unique solution is given by.

$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

Example: We shall solve

$y + z = 2$

$x + z = 2$

$x + y = 2$

The system can be written in the form:

$$\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

Here one can check that $|A| = 2$, hence the matrix is nonsingular and therefore invertible. The inverse is given by

$$A^{-1} = \frac{1}{2} Adj\left(\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\1 & 1 & -1\end{bmatrix}^T\right)$$

$$= \begin{bmatrix}-1/2 & 1/2 & 1/2\\1/2 & -1/2 & 1/2\\1/2 & 1/2 & -1/2\end{bmatrix}$$

Hence, $\begin{bmatrix}x\\y\\z\end{bmatrix} = A^{-1}b = \begin{bmatrix}-1/2 & 1/2 & 1/2\\1/2 & -1/2 & 1/2\\1/2 & 1/2 & -1/2\end{bmatrix}\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Thus $(x, y, z) = (1,1,1)$ is the unique solution.

When $n > m$, in general the system is underdetermined and by elementary row and column operations m of the variables of the solutions can be described in terms of the remaining $n - m$ variables.

Example: Suppose we are given,

$x - y + z = 1$

$x + y + z = 1$

Then we have the number of variables $n = 3 > 2 = m$ the number of equations. One immediately finds the infinite number of solutions as:

$\{(x, 0, 1 - x) \mid x \in \mathbb{R}\}$

or $\{(1 - z, 0, z) \mid z \in \mathbb{R}\}$

When $n < m$, in general, the system is over-determined and there is no solution.

Example: Suppose,

$x + y = 2$

$x - y = 2$

$2x + y = 3$

Then we have the number of variables $n = 2 < 3 = m$ the number of equations. One immediately observes that the first two equations give unique solution as $x = 2$ □nd $y = 0$. But these values don't satisfy the third equation. Hence the system is inconsistent.

5. Application:

Suppose in a market there are n commodities. The price of i^{th} commodity is P_i. Moreover demand and supply of i^{th} commodity are linear functions given by:

$Q_{di} = a_{i0} + a_{i1}P_1 + \cdots a_{in}P_n$

$Q_{si} = b_{i0} + b_{i1}P_1 + \cdots b_{in}P_n$

The Market Equilibrium demands that there is no excess demand i.e. $E_i \equiv Q_{di} - Q_{si} = 0$. This leads to system of linear equations for the variables $P_1, P_2, \dots P_n$. This system of equations can be solved by the methods described in this module.

Now, let us summarize the topics that we have gone through.

6. Summary:

Through appropriate examples of matrices we concluded that unlike in case of real numbers, matrix multiplication is not commutative and also we have seen that product of two matrices may be a zero matrix without any of the matrix being a zero matrix. Similarly “cancellation law also does not hold” is shown by appropriate example. We then have covered the discussion on determinant, Adjoint and Inverse of a Matrix. We have profusely illustrated our discussion through examples. We then returned to the thing with which we started our study of Matrices viz. to simultaneous linear equations. Here also we considered several simple examples to illustrate the points that we make. As an illustration of application, we have considered multi-commodity linear model and discussed how solution to Market Equilibrium of these commodities can be achieved through solving simultaneous linear equations.