[Glossary]

Vector Spaces and Matrices

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Glossary

Vector space or Linear Space

Vector space V over a Field F which is usually taken as the field of Real numbers or the field of complex numbers is a set together with operations called addition and scalar multiplications having the properties:

(i) For each $x,y \in V$ and $\alpha \in F$, $x+y \in V$ and $\alpha x \in V$.

(ii) For each $x,y,z \in V$,

$$x+(y+z)=(x+y)+z.$$

This property is called associativity.

(iii) There exists element $0 \in V$ with the property that $x+0=0+x=x$.

(iv) For each element $x \in V$ there exists element which is denoted by $-x \in V$ such that $x+(-x)=0$.

(v) For all $x,y \in V$, $x+y=y+x$

(vi) For all $x \in V$ and $\alpha,\beta \in F$ $(a\beta)(x)=a(\beta x)=(\beta a)(x)$

(vii) For all $x,y \in V$ and $\alpha \in F$ $a(x+y)=ax+ay$

(viii) For all $x \in V$ and $\alpha,\beta \in F$ $(a+\beta)x=ax+\beta x$

(ix) $1x=x$ for all $x \in V$

Linear Combination

If $v^1,v^2,\ldots,v^m$ are elements of a vector space $\mathbb{R}^n$, a vector

$v=\alpha_1 v^1+\alpha_2 v^2+\cdots+\alpha_m v^m$

where $\alpha_1,\alpha_2,\ldots\alpha_m$ are $scalars$ that is they are real numbers, is called a linear combination of vectors $v^1,v^2,\ldots,v^m$

Linearly Independent set

A set $\{v^1, v^2, \ldots, v^m\}$ is a called linearly independent set if zero can be expressed only as a trivial linear combination of these vectors. In other words no non-trivial linear combination of these vectors is the zero vector. That is

If $\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_m v^m = 0$ then $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$

Span

If $S \subseteq \mathbb{R}^n$ is a given set, its linear span or simply span denoted by $[S]$ is defined as the set of all the linear combinations of all the elements of S. i.e.

$$[S] = \{\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_m v^m \,|\, v^1, v^2, \ldots, v^m \in S \;\; and\; \alpha_1, \alpha_2, \ldots \alpha_m \in \mathbb{R}\}$$

Basis

A subset of $\mathbb{R}^n$ is said to a basis for $\mathbb{R}^n$ if it is a linearly independent set and it also spans the whole space $\mathbb{R}^n$.

Scalar Product or Inner Product

Suppose $x = (x_1, x_2, \ldots x_n)\; and\; y = (y_1, y_2, \ldots y_n) \in \mathbb{R}^n$. Then scalar product of $x\; and\; y$ which is denoted by $x \cdot y$ or $<x, y>$ is defined as:

$$x \cdot y = < x, y > = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

Orthogonal vectors

Two vectors $x, y \in \mathbb{R}^n$ are said to be orthogonal if

$$x \cdot y = < x, y > = 0.$$

Matrix

Matrix is simply an array of entities usually of numbers and variables. A general matrix A of m rows and n columns is written as:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad i = 1,2 \dots m \text{ and } j = 1,2, \dots n$$

Often it is denoted by: $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$

Addition of matrices

When A and B are two matrices with real numbers or complex numbers as entries and of the same type say having m rows and n columns that is they are $m \times n$, then they are added as:

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

That is ith row, jth column element of the first matrix is added to the ith row, jth column of the second matrix to form the ith row, jth column entry of the matrix $A + B$.

Multiplication of a scalar with a Matrix i.e. scalar multiplication

If a matrix $A = [a_{ij}]$ and $\alpha \in \mathbb{R}$ are given, we define scalar multiplication as:

$$\alpha A = [\alpha a_{ij}]$$

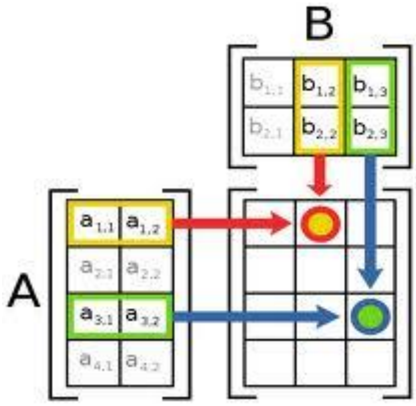
which means that α is multiplied to every entry of the matrix A.

Multiplication of Two matrices

Suppose we are given two matrices A and B. The first matrix A is $m \times k$ and the second matrix B is $k \times n$. Then the product AB is the $m \times n$ matrix whose general term c_{ij} is formed by multiplying the corresponding terms of the ith row of A and jth column of B both of which are having k entries and then summing up these multiplied term. That is,

$$C = AB = [c_{ij}] \;\; where\; c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$

A special case can be visualized as:



Zero Matrix And Identity Matrix

A matrix $A=[a_{ij}]$ of the type $m \times n$, all of whose entries are zero, is called a zero matrix.

A matrix $A=[a_{ij}]$ of the type $n \times n$, that is a matrix with equal number of rows and columns is called a square matrix.

For a square matrix $A=[a_{ij}]$ of the type $n \times n$, the terms a_{ii}, $i = 1,2,\dots n$; are called main diagonal term. If in a square matrix all terms on the main diagonal are identity i.e. 1 and off the main diagonal all the terms are zero, then the matrix is called the Identity matrix, this matrix is usually denoted by I. That is

$$I = [a_{ij}] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad i = 1,2\dots n \; and\; j = 1,2,\dots n$$

which can also be written as:

$I=[a_{ij}]$ where $a_{ii} = 1$, $i = 1,\dots n$ and $a_{ij} = 0 \quad i \neq j,\; i,j = 1,\dots n$. This matrix has interesting property viz.

$$AI = IA = A$$

Transpose of a Matrix

Transpose of a $m \times n$ matrix $A = [a_{ij}]$ is a $n \times m$ matrix $A^T = [a_{ji}]$ that means jth row, i^{th} column entry in A^T is the i^{th} row, j^{th} column entry of A. In other words the rows of A are made columns of A^T

Symmetric Matrix

A $n \times n$ square matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ is called symmetric if $\boldsymbol{a_{ij}} = \boldsymbol{a_{ji}}$ for all $i, j = 1,2 \dots n$. That is A is symmetric if $\boldsymbol{A} = \boldsymbol{A^T}$.

Orthogonal Matrix

A $n \times n$ square matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ is called an orthogonal matrix if $\boldsymbol{AA^T} = \boldsymbol{I}$.

Idempotent Matrix

A $n \times n$ square matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ is called idempotent if $\boldsymbol{A^2} = \boldsymbol{A}$.