[Academic Script]

[Extreme Value & power series Representation of Function]

Subject:	Business Economics
Course:	B.A., 2nd Semester, Undergraduate
Paper No. & Title:	Paper – 202 Mathematics for Business Economics
Unit No. & Title:	Unit - 3 Single variable Differentiation
Lecture No. & Title:	1: Extreme Value & power series Representation of Function

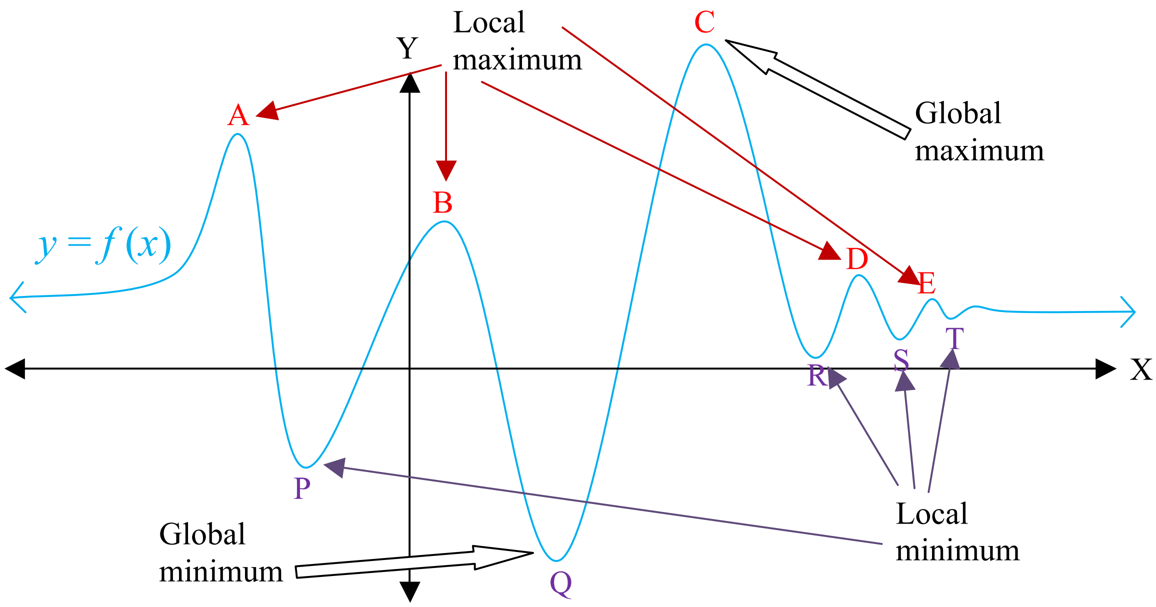
In this module, we mainly discuss about the applications of derivatives in finding the high nodes, the low nodes and the stationary nodes of any differentiable curve. Power Functions and polynomials are the most regularly used and easier to deal with, type of functions for most purpose. This very fact provides us enough motivation to talk about the power series representation of a function.

Broadly speaking, today we will discuss the following points, as part of applications of differentiation in Unit-3.

- Finding maxima/minima
- Finding stationary points
- Finding points of inflexion
- Generating Taylor's series and Maclaurin's series and
- Finding Marginal cost and Marginal revenue

Now let's discuss Extreme values:

Extreme Values:
The maxima and minima of a function, known collectively as extrema are the largest and the smallest values of the function, either within a given range (i.e. the local extrema) or on the entire domain of the function (i.e. the global extrema or absolute extrema).



For example, see the plot of a function $y = f(x)$. There are many high nodes and low nodes available in it. We say a node (or a point) is a high node if the graph falls everywhere in its local neighbourhood. Such nodes are called points of local maxima.

Similarly,

we say a node (or a point) is a low node if the graph rises everywhere in its local neighbourhood. Such nodes are called points of local minima.

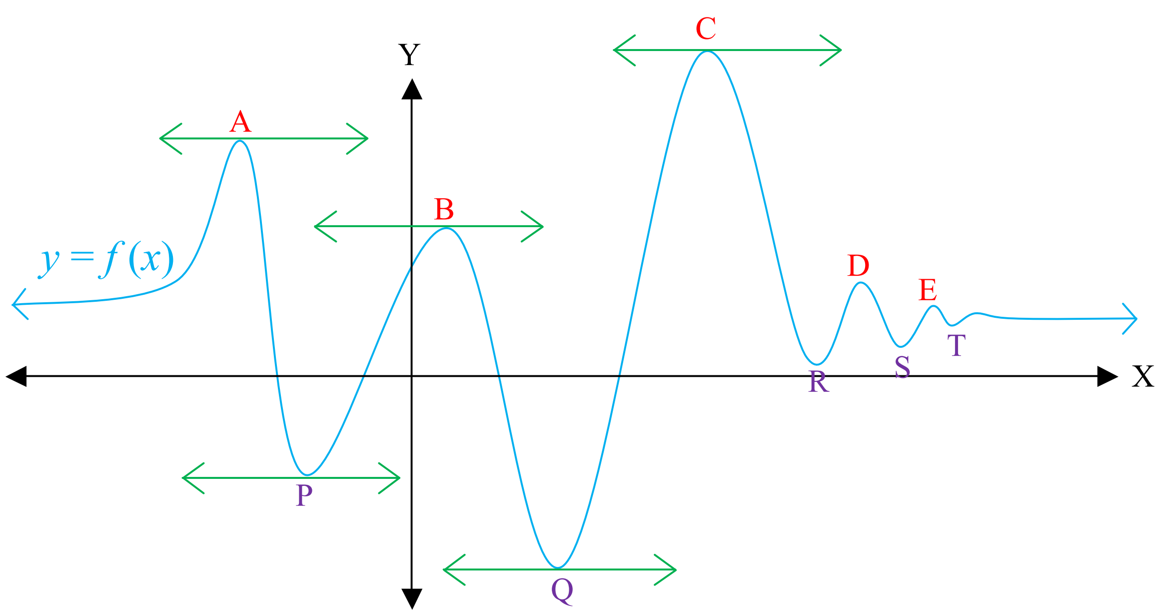
These phenomena can be easily observed for some of the nodes available in the plot.

$\therefore$ Points A, B, D and E are the points of local maximum and the point C is the point of global maximum.

Similarly,

Points P, R, S and T are the points of local minimum and the point Q is the point of global minimum.

<u>Stationary Point</u>: A stationary point of a function $f : A \to R$ of one variable is a point $a \in A$ for which $f^{'}(a) = 0$ (equivalently, we can also say that it is a point on the curve where the tangent is parallel to X-axis). It is also called a critical point of a function.



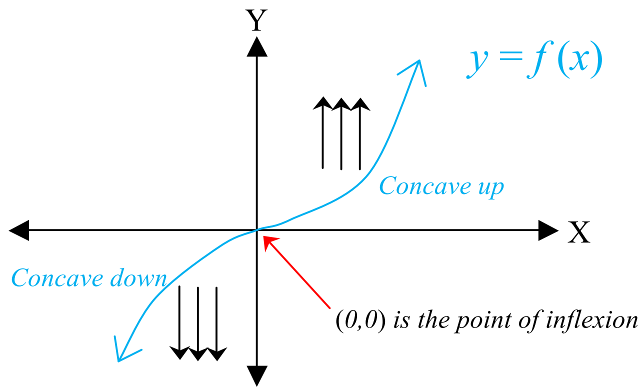
For example, see the plot of a function $y = f(x)$.

The points A, B, C, P, Q etc. are all stationary points, because the tangents at these points are parallel to X-axis.

This clearly suggests that every point of extrema of a function is also a stationary point of it.

Now let's understand the points of inflexion.

Point of Inflexion:
Inflexion points are the points on the curve at which the curve changes from being concave down to concave up or vice versa.



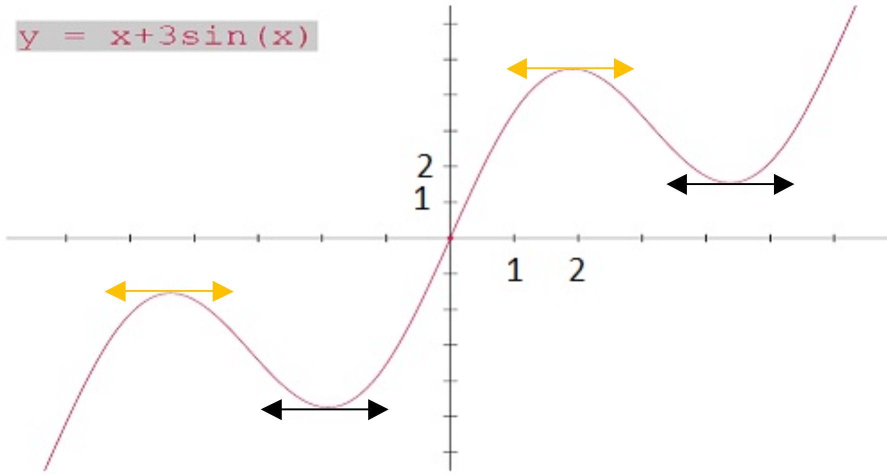
For example, see the plot of a function $y = f(x)$.
Here the curve is increasing and concave down (i.e. looking down) on the left of origin and on the right hand side of origin the curve is still increasing but concave up (i.e. looking up). This means that the curve $y = f(x)$ changes its concavity at the origin (i.e. (0,0)).
$\therefore$ (0,0) is the point of inflexion of above curve.

Hence,
by looking at the graph one can easily point out the maxima and minima of the corresponding function.

But, this graphical approach of getting the maxima and minima is not feasible always, because of the following reasons:

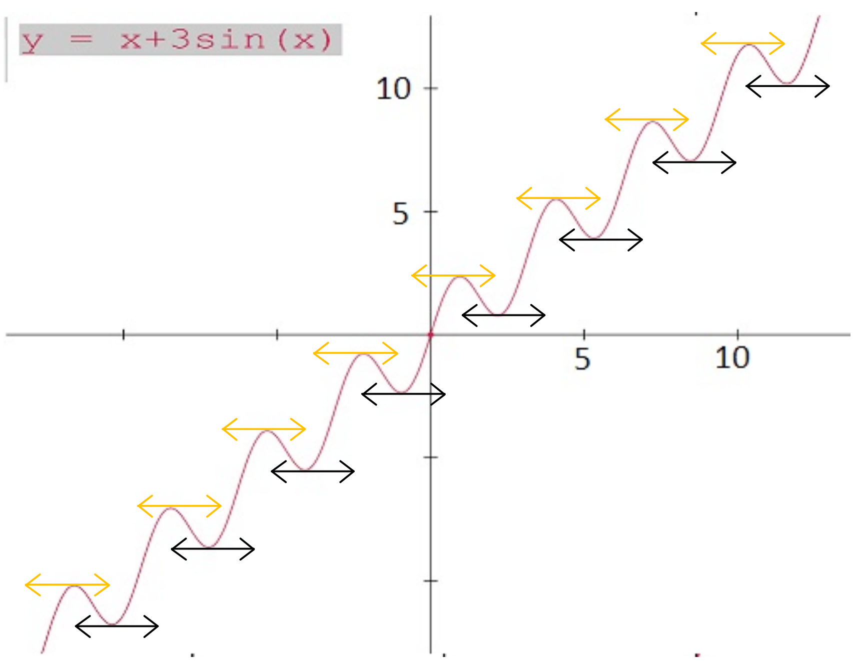
1. In most graphs, we can only predict those local maxima and local minima which are clearly visible in the plot. Whereas the maxima and minima points of function that have not been drawn are missed.

To understand this, see the graph of $y = x + 3\sin x$



By looking at this graph, it seems as if there are just 2 points of local maxima and 2 points of local minima of function $y = x + 3\sin x$.

Now let's see the following graph of $y = x + 3\sin x$ with different scaling...



By looking at this graph, it is clear that there are more than 2 points of local maxima and also more than 2 points of local minima of function $y = x + 3\sin x$.

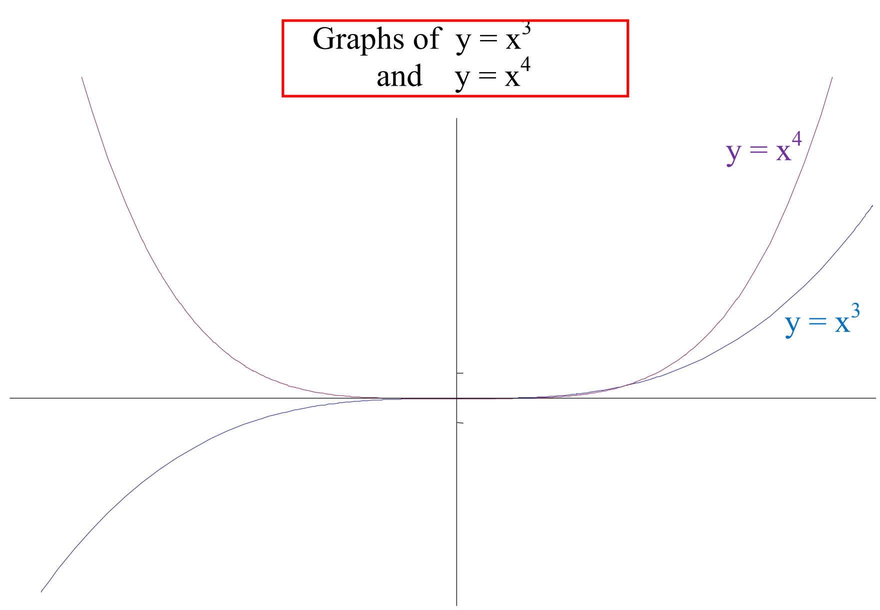
2. It's not an accurate way to estimate the exact values of maxima or minima numerically, as the graphs only illustrate approximate location where the maxima or minima will occur.

This motivates the need to have some better alternative of getting the points of local maxima or local minima of a function.

So let's discuss a method which uses the concept of derivatives to obtain points of maxima or minima.

Working Rules to find Maxima or Minima of $y = f(x)$

1. First differentiate the function to get $f'(x)$.
2. Solve $f'(x) = 0$ to get all its stationary points say x_1 and x_2.
3. Again differentiate the function to get $f''(x)$.
4. Now evaluate f'' at the above stationary points and conclude as under:
   (i) (x_1, y_1) is a point of local minima if $f''(x_1) > 0$
   (ii) (x_1, y_1) is a point of local maxima if $f''(x_1) < 0$
   (iii) If $f''(x_1) = 0$ then the point (x_1, y_1) is neither a point of maxima nor a point of minima and it may or may not be the point of inflexion.
   This can be easily seen from the graphs of functions $y = x^3$ and $y = x^4$.



For the functions $f(x)=x^3$ and $g(x)=x^4$, the second derivative at $x=0$ is 0 i.e. $f''(0)=0$ and $g''(0)=0$ but from the graphs it can be easily understood that the curve $f(x)=x^3$ changes its concavity at the point $(0,0)$. This means that the origin is the point of inflexion of curve $f(x)=x^3$ whereas the curve $g(x)=x^4$ remains concave up on both sides of Y-axis i.e. it does not change its concavity at point $(0,0)$, so it's NOT a point of inflexion of $g(x)=x^4$.

After having learnt these concepts of high nodes and low nodes of a graph, let's solve some questions now.

Example: 1) Find all the stationary points of $y=x^3-3x^2-9x$ and also check the nature of function at these points.
Solution:

Here $y=x^3-3x^2-9x$

$\therefore \dfrac{dy}{dx}=3x^2-6x-9=3(x^2-2x-3)=3(x-3)(x+1)$

On solving $\dfrac{dy}{dx}=0$ we get, $3(x-3)(x+1)=0$.

i.e. $x=3$ or $x=-1$

$\therefore\ x=3,\,-1$ are the two stationary points of $y=x^3-3x^2-9x$.

Now we will check the nature of function at these stationary points.

$\therefore \dfrac{d^2y}{dx^2}=6x-6=6(x-1)$.

$\therefore \left(\dfrac{d^2y}{dx^2}\right)_{x=3}=6(3-1)=12>0$ and $\left(\dfrac{d^2y}{dx^2}\right)_{x=-1}=6(-1-1)=-12<0$

So as per the second derivative test, the given function has Local Minima at $x=3$ and Local Maxima at $x=-1$.

Example: 2) Find the extreme points of $y=x^3+7x$, if any.
Solution:

Here $y=x^3+7x$

$\therefore \dfrac{dy}{dx}=3x^2+7$

On solving $\dfrac{dy}{dx}=0$ we do not get any real solutions.

$\therefore$ The function $y=x^3+7x$ do not have any stationary points.

Hence the function $y = x^3 + 7x$ does not have any maxima or minima points.

Example: 3) Find a local maximum value attained by the function $f(x, y) = x^3 + 8y^3 - 6xy$ subject to $2x + 5y = 10$.
Solution:

We have to solve $f(x, y) = x^3 + 8y^3 - 6xy$ subject to condition $2x + 5y = 10$.
Using the above condition we first rewrite a more simplified new function to be maximised.

For this consider $w = f(x, y)$ subject to $y = \frac{10 - 2x}{5}$.

i.e. $w = f\left(x, \frac{10 - 2x}{5}\right)$

i.e. $w = x^3 + 8\left(\frac{10 - 2x}{5}\right)^3 - 6x\left(\frac{10 - 2x}{5}\right)$

i.e. $w = \frac{1}{125}\left[125x^3 + 8(10 - 2x)^3 - 150x(10 - 2x)\right]$

i.e. $w = \frac{1}{125}\left[61x^3 + 620x^2 - 3100x + 8000\right] := g(x)$

Now our aim is to get point of maxima of above function $w = g(x)$.

$\therefore \frac{dw}{dx} = g'(x) = \frac{1}{125}\left[183x^2 + 1240x - 3100\right]$

On solving $g'(x) = 0$ we get,

$x = \frac{-1240 + 1951.10}{2 \times 183}$ or $x = \frac{-1240 - 1951.10}{2 \times 183}$

i.e. $x = 1.943$ or $x = -3.392$ approximately.

Now we will check the nature of function $w = g(x)$ at these stationary points.

$\therefore \frac{d^2w}{dx^2} = g''(x) = \frac{1}{125}\left[366x + 1240\right] = \frac{2}{125}\left[183x + 620\right]$

$\therefore g''(1.943) = 15.609 > 0$ and $g''(-3.392) = -9.421 \times 10^{-5} < 0$.

So as per the second derivative test, $w = g(x)$ has a local minima at $x = 1.943$ and local maxima at $x = -3.392$.

Since we are only interested in a local maxima of $w = f(x, y)$.

$\therefore$ For $x = -3.392$, the corresponding value of y will be $y = 3.357$.

$\therefore$ The required point of local maxima is $(x, y) = (-3.392, 3.357)$ and the corresponding local maximum value of $f(x, y) = 331.947$ approximately.

Now, let's discuss another useful application namely Taylor's series and Maclaurin's series.

<u>Taylor's series and Maclaurin's series:</u>

Let $f : R \to R$ be a real valued function of real variable. If f is infinitely many times differentiable at $x = a$ then the Taylor's series of f at a is given by $f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2 \frac{f''(a)}{2!} + (x-a)^3 \frac{f'''(a)}{3!} + \ldots + (x-a)^r \frac{f^{(r)}(a)}{r!} + \ldots$

i.e. $f(x) = \sum_{r=0}^{\infty} (x-a)^r \frac{f^{(r)}(a)}{r!}$

Here, if value of $a = 0$ then the above series can be re-written as $f(x) = f(0) + x \frac{f'(0)}{1!} + x^2 \frac{f''(0)}{2!} + x^3 \frac{f'''(0)}{3!} + \ldots + x^r \frac{f^{(r)}(0)}{r!} + \ldots$

i.e. $f(x) = \sum_{r=0}^{\infty} x^r \frac{f^{(r)}(0)}{r!}$

This is called Maclaurin's series of function $f(x)$.

Hence, it turns out that the Maclaurin's series is a particular case of Taylor's series.

Example: 4) Write the Maclaurin's series expansion of $f(x)=(x-1)^3-3x+10$.

Solution:

Here $y=f(x)=(x-1)^3-3x+10 \quad \Rightarrow \quad f(0)=9$

$\therefore \frac{dy}{dx}=f'(x)=3(x-1)^2-3 \quad \Rightarrow \quad f'(0)=0$

$\therefore \frac{d^2y}{dx^2}=f''(x)=6(x-1) \quad \Rightarrow \quad f''(0)=-6$

$\therefore \frac{d^3y}{dx^3}=f'''(x)=6 \quad \Rightarrow \quad f'''(0)=6$

$\therefore \frac{d^ny}{dx^n}=f^{(n)}(x)=0$ for all $n\geq 4 \quad \Rightarrow \quad f^{(n)}(0)=0$ for all $n\geq 4$.

The Maclaurin's series expansion of $y=f(x)$ is given by

$$f(x)=f(0)+x\,\frac{f'(0)}{1!}+x^2\,\frac{f''(0)}{2!}+x^3\,\frac{f'''(0)}{3!}+\ldots+x^r\,\frac{f^{(r)}(0)}{r!}+\ldots$$

$$=(9)+x\,\frac{(0)}{1!}+x^2\,\frac{(-6)}{2!}+x^3\,\frac{(6)}{3!}+x^4\,\frac{(0)}{4!}+\ldots+x^r\,\frac{(0)}{r!}+\ldots$$

$$=9-3x^2+x^3$$

Hence, the required Maclaurin's series of $f(x)$ is $9-3x^2+x^3$.

Example: 5) Write the Taylor's series expansion of $f(x)=(x-1)^3-3x+10$ about the point $x=3$.

Solution:

Let $y=f(x)=(x-1)^3-3x+10 \quad \Rightarrow \quad f(3)=9$

$\therefore \frac{dy}{dx}=f'(x)=3(x-1)^2-3 \quad \Rightarrow \quad f'(3)=9$

$\therefore \frac{d^2y}{dx^2}=f''(x)=6(x-1) \quad \Rightarrow \quad f''(3)=12$

$\therefore \frac{d^3y}{dx^3}=f'''(x)=6 \quad \Rightarrow \quad f'''(3)=6$

$\therefore \frac{d^ny}{dx^n}=f^{(n)}(x)=0$ for all $n\geq 4 \quad \Rightarrow \quad f^{(n)}(3)=0$ for all $n\geq 4$.

The Maclaurin's series expansion of $y=f(x)$ is given by

$$f(x) = f(3) + (x-3)\frac{f'(3)}{1!} + (x-3)^2\frac{f''(3)}{2!} + (x-3)^3\frac{f'''(3)}{3!} + (x-3)^4\frac{f^{(4)}(3)}{4!} + \ldots + (x-3)^r\frac{f^{(r)}(3)}{r!} + \ldots$$

$$= 9 + (x-3)\frac{(9)}{1!} + (x-3)^2\frac{(12)}{2!} + (x-3)^3\frac{(6)}{3!} + (x-3)^4\frac{(0)}{4!} + \ldots + (x-3)^r\frac{(0)}{r!} + \ldots$$

$$= 9 + 9(x-3) + 6(x-3)^2 + (x-3)^3$$

Hence, the required Taylor's series expansion is $f(x) = 9 + 9(x-3) + 6(x-3)^2 + (x-3)^3$.

Example: 6) Write the Maclaurin's series expansion of $f(x) = \sin x$

Solution:

Here $y = f(x) = \sin x \quad \Rightarrow \quad f(0) = 0$

$\therefore \frac{dy}{dx} = f'(x) = \cos x \quad \Rightarrow \quad f'(0) = 1$

$\therefore \frac{d^2y}{dx^2} = f''(x) = -\sin x \quad \Rightarrow \quad f''(0) = 0$

$\therefore \frac{d^3y}{dx^3} = f'''(x) = -\cos x \quad \Rightarrow \quad f'''(0) = -1$

$\therefore \frac{d^4y}{dx^4} = f^{(4)}(x) = \sin x \quad \Rightarrow \quad f^{(4)}(0) = 0$

$\therefore \frac{d^5y}{dx^5} = f^{(5)}(x) = \cos x \quad \Rightarrow \quad f^{(5)}(0) = 1$

and so on.

The Maclaurin's series expansion of $y = f(x)$ is given by

$$f(x) = f(0) + x\,\frac{f'(0)}{1!} + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \ldots$$

$$= (0) + x\,\frac{(1)}{1!} + x^2\frac{(0)}{2!} + x^3\frac{(-1)}{3!} + x^4\frac{(0)}{4!} + x^5\frac{(1)}{5!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Hence, the Maclaurin's series of $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$.

Now let's talk about some straight forward applications of differentiation in Economics.

Applications to Economics

Being the students of economics,
I assume you all to be well versed with the terminologies like cost, revenue, profit, total cost, total revenue, marginal revenue etc.
But just for the sake of completeness, let's quickly go through some definitions.

Total Revenue:
It can be calculated as the selling price of the product times the quantity sold.
$\therefore TR = P \times Q$ where P is the per unit price of product

Marginal Revenue:
Marginal revenue is equal to the ratio of the change in revenue for some change in quantity sold to that change in quantity sold.
$\therefore MR = \dfrac{\Delta R}{\Delta Q}$
i.e. It's the additional revenue generated by increasing the product sales by 1 unit.

Marginal cost:
Marginal cost is the change in the total cost that arises when the quantity produced is incremented by 1 unit.
i.e. it is the cost of producing 1 more unit of a good.
$\therefore MC = \dfrac{\Delta C}{\Delta Q}$ i.e. the ratio of change in variable cost to change in the quantity of goods.

Marginal Profit:
It's the difference between the marginal revenue and the marginal cost of producing one additional unit of output.
$\therefore M_{\pi} = MR - MC$ where M_{π} denotes marginal profit.

Marginal Profit Maximisation:
To maximise marginal profit a firm should continue to produce a good or a service up to the point where marginal profit is zero.

Note that, if the cost function is a differentiable curve and the per unit change in quantity is infinitesimally small, then the marginal revenue and marginal cost can be re-defined in terms of derivatives as under:

Marginal revenue is given by $MR = \frac{dR}{dQ}$ and

Marginal cost is given by $MC = \frac{dC}{dQ}$

Example: 7) If the total cost function is given by $C = q^3 - 5q^2 + 14q + 75$, where q represents the quantity of goods, then obtain the marginal cost function.

Solution:

Total cost function $C = q^3 - 5q^2 + 14q + 75$ and the marginal cost is given by $MC = \frac{dC}{dq}$.

$$\therefore MC = \frac{d}{dq}\left(q^3 - 5q^2 + 14q + 75\right)$$

$$\therefore MC = 3q^2 - 10q + 14$$

So the required marginal cost function is $MC = 3q^2 - 10q + 14$.

Example: 8) If the average cost function is given by $AC = q^2 - 4q + 214$, where q represents the quantity of goods, then calculate the marginal cost function and also find the rate of change of average cost.

Solution:

$AC = q^2 - 4q + 214$ is the average cost function.

We know that $TC = q \cdot AC$

$$\therefore TC = q \cdot \left(q^2 - 4q + 214\right)$$

$$\therefore TC = q^3 - 4q^2 + 214q$$

Now, $MC = \frac{d}{dx}\left(TC\right)$

$$\therefore MC = \frac{d}{dq}\left(q^3 - 4q^2 + 214q\right)$$

$$\therefore MC = 3q^2 - 8q + 214$$

And the rate of change of average cost $= \frac{d}{dq}(AC)$

$$= \frac{d}{dq}\left(q^2 - 4q + 214\right)$$

$$= 2q - 4$$

So the required marginal cost function is $MC = 3q^2 - 8q + 214$ and the function describing the rate of change of average cost is $2q - 4$.

Example: 9) If the average revenue function is given by $AR = 60 - 3q$, where q represents the quantity of goods, then calculate the marginal revenue function.
Solution:

The average revenue function is $AR = 60 - 3q$.
We know that $TR = q \cdot AR$

$$\therefore TR = q \cdot (60 - 3q)$$

$$= 60q - 3q^2$$

Now, $MR = \frac{d}{dq}(TR)$

$$\therefore MR = \frac{d}{dq}\left(60q - 3q^2\right)$$

$$= 60 - 6q$$

So the required marginal revenue function is $60 - 6q$.

Now,
Let’s summarise everything that we have done.
In this module, we discussed the concept of maxima and minima for a real valued single variable function along with its point of inflexion, if any. Also we discussed the procedure to obtain power series representation of any suitable function.
In addition, we continued our talk to include a few applications of derivatives such as the concept of marginal cost, marginal revenue etc.

With this I end up this module.
Hope you liked and understood the topics covered here.