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Limit, Continuity & Differentiability of a real valued function of a real variable

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1. Introduction

Friends, we meet here to discuss the two important concepts of calculus and these are continuity and differentiability. These two concepts rely on the concept of limit and so first we make an attempt to understand the concept of limit of a real valued function.

2. Limit of a real-valued function

Let $f(t)$ be a real-valued function which is defined on some open interval I containing a point x. We say that $f(t)$ has limit l as t tends to x or $f(t)$ has limit l at $t = x$, and write,

$\lim_{t \to x} f(t) = l,$ if for every $\epsilon > 0$ there exists $\delta > 0$, such that $|f(t) - l| < \epsilon$, whenever $t \in I \text{ and } 0 < |t - x| < \delta$.

Before we proceed, it is important to gather some important points related to the definition of limit.

1. We may write $\lim_{t\to x} f(t) = l$ in the form $f(t) \to l$ as $t \to x$.
2. Although, in the definition of limit of a function at a point x, we have assumed that the function $f(t)$ be defined on an interval containing the point x, it is not absolutely necessary to assume that $f(t)$ is defined at the point x. The inequality $0 < |t - x| < \delta$ in the definition of limit clearly suggests that while evaluating the limit of $f(t)$ at the point x, the value of the function $f(t)$ at $t = x$ has no role to play.
3. Limit of a function may or may not exist at a given point.
4. If for a function $f(t)$, its limit exists at $t = x$, then it is unique.

Now let us try to understand the concept of limit geometrically.

Consider the function $s = f(t)$ for which we have $\lim_{t \to x} f(t) = l$. Geometrically, this means that for any arbitrarily thin horizontal strip H containing the line $s = l$, we should be able to find an interval I containing the point x such that the graph of $s = f(t)$ for $t \in I - \{x\}$ lies completely in the horizontal strip H.

We now consider some illustrations about evaluating the limit of a function at a given point.

Example: Evaluate $\lim_{t \to 2} t^2$.

Solution: Here as the values of t get closer and closer to 2, the function t^2 gets closer and closer to the value 4. Thus, we believe that the required limit is 4. To prove this, we consider an arbitrarily small $\epsilon > 0$ and determine a $\delta > 0$, such that

$|t^2 - 4| < \epsilon \; whenever \; 0 < |t - 2| < \delta.$

Choosing $\delta = \frac{\epsilon}{5} < 1$, we see that

$|t^2 - 4| = |t - 2||t + 2| < \frac{\epsilon}{5}|t + 2| < \epsilon$ whenever $0 < |t - 2| < \delta = \frac{\epsilon}{5}$. Thus $\lim_{t \to 2} t^2 = 4$.

In general, it can be shown that $\lim_{t \to c} t^2 = c^2$ for every real number c. this may tempt us to believe that limit of a function $f(t)$ at $t = c$ can be derived by inserting $t = c$ in $f(t)$. but this is not always true as seen in the following example.

Example: Evaluate $\lim_{t \to 1} \frac{t^2 + t - 2}{t - 1}$.

Solution: Here notice that the limit cannot be obtained by inserting $t = 1$ in the given function because it gives us an undefined quantity. For evaluating the limit in this case, first we observe that

$$\lim_{t \to 1} \frac{t^2 + t - 2}{t - 1} = \lim_{t \to 1} \frac{(t - 1)(t + 2)}{(t - 1)} = \lim_{t \to 1} (t + 2).$$

It may now be verified that the required limit is 3.

Next, we consider an example where the limit does not exist.

Example: For the function $f(t)=\begin{cases}-1, & t<0\\ 1, & t\geq 0\end{cases}$ show that its limit does not exist at $t=0$.

Solution: If possible, suppose $\lim_{t\to 0} f(t)=l$. then $\min\{|1-l|,|1+l|\}>0$.

Now if $\lim_{t\to 0} f(t)=l$ then for $\epsilon=\frac{1}{2}\min\{|1-l|,|1+l|\}$, there exists $\delta>0$ such that $|f(t)-l|<\epsilon$, whenever $0<|t|<\delta$. But $|f(t)-l|$ is either $|1-l|\ or |1+l|$ and both these values are at least 2ϵ. So we arrive at a contradiction.

Sometimes when the domain of a function is an interval of the type (a,b), we cannot talk about the limit of this function at the end points of this interval. However, we may investigate about the one-sided limits at these endpoints which are sometimes useful in understanding the behavior of the function at the endpoints.

3. The concept of One-sided limits

Let $f(t)$ be a function defined on an interval just to the right of x. We say that $f(t)$ has l as a limit from right or right limit l at $t=x$, and write, $\lim_{t\to x^+} f(t)=l$, if for every $\epsilon>0$ there exists $\delta>0$ such that $|f(t)-l|<\epsilon$, whenever $x<t<x+\delta$.

Similarly, if $f(t)$ is defined on an interval just to the left of x then we say that $f(t)$ has left limit l at $t=x$, and write,

$\lim_{t\to x^-} f(t)=l$, if for every $\epsilon>0$ there exists $\delta>0$ such that $|f(t)-l|<\epsilon$, whenever $x-\delta<t<x$.

We saw that the function $f(t)=\begin{cases}-1, & t<0\\ 1, & t\geq 0\end{cases}$ does not have limit at $t=0$. But it can be easily verified that

$\lim_{t\to 0^+} f(t)=1$ and $\lim_{t\to 0^-} f(t)=-1$.

4. Basic properties of Limits

Suppose $\lim_{t\to x} f(t) = l_1$ and $\lim_{t\to x} g(t) = l_2$. Then

1. $\lim_{t\to x}\{f(t) \pm g(t)\} = l_1 \pm l_2$.
2. $\lim_{t\to x}\{f(t)g(t)\} = l_1 l_2$.
3. $\lim_{t\to x}\{f(t)/g(t)\} = \frac{l_1}{l_2}$, provided that $l_2 \neq 0$.

Note that the first two properties also hold if we have any finite number of functions instead of two. These properties are very useful in the evaluation of limits of certain functions and sometimes in proving general results like the following theorem.

Theorem: If $p(t)$ is a polynomial function then $\lim_{t\to a} p(t) = p(a)$ for every real number a. Further, for any rational function $\frac{p(t)}{q(t)}$, if $q(a) \neq 0$, then $\lim_{t\to a} \frac{p(t)}{q(t)} = \frac{p(a)}{q(a)}$.

We saw that for polynomial functions and rational functions, the limit at a given point of their domain is simply the value of the respective function at that point. There are many more functions for which this holds and this leads us to the definition of continuous functions.

5. Continuity of a function at a point

Let $f(t)$ be a real-valued function defined on some subset D of $\mathbb{R}$. We say that $f(t)$ is continuous at the point $x \in D$ if the domain set D contains an open interval containing the point x, and further $\lim_{t\to x} f(t) = f(x)$. If $f(t)$ is continuous at each and every point of the domain set D then we say that f is continuous on D.

If a function $f(t)$ fails to be continuous at a point x, then we say that $f(t)$ is discontinuous at x.

From the definition of continuity of a function at a point, we infer that for a function f to be continuous at x, three things should happen.

1. The function f should be defined at x.

2. The function f should have limit at x.

3. The limit of the function f at x should be equal to the value of the function f at x.

Since these three things happen for polynomial functions everywhere, all polynomial functions are examples of continuous functions on$\mathbb{R}$.

6. **The concept of right and left continuity**

A function $f(t)$ is continuous at $t = x$ from right or left, if $\lim_{t \to x^+} f(t) = f(x)$ or $\lim_{t \to x^-} f(t) = f(x)$ respectively.

Remark: Obviously**,** f is continuous at x if and only if it is continuous from right as well as left.

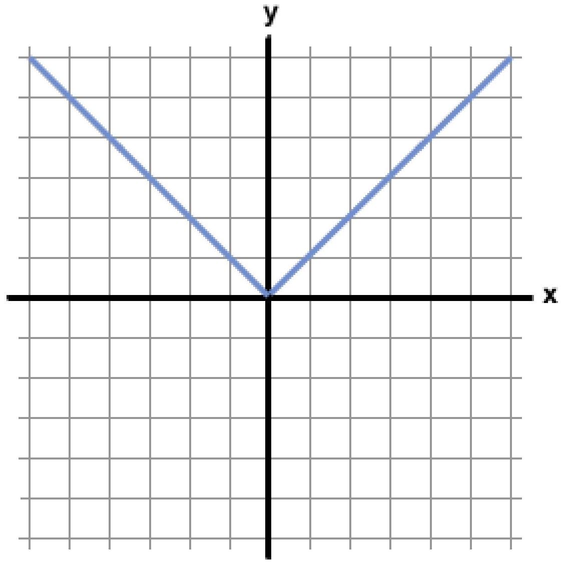
Example: Prove that $f(t) = |t|$ is continuous on $\mathbb{R}$.

Solution: Let x be an arbitrary real number. In order to show that the function $|t|$ is continuous at x, we need to show that $\lim_{t\to x}|t| = |x|$. Let $\epsilon > 0$ be given. Choosing $\delta = \epsilon$, we see that $\big||t| - |x|\big| < \epsilon$ whenever $|t - x| < \delta$ because of the well-known inequality

$\big||t| - |x|\big| \le |t - x|$.

Sometimes it is easy to judge the continuity of a function using its graph. If we can draw the graph of a function f in an open interval without lifting the pen then such a function is continuous at each and every point of this interval.

For instance if we think about the graph of $|t|$

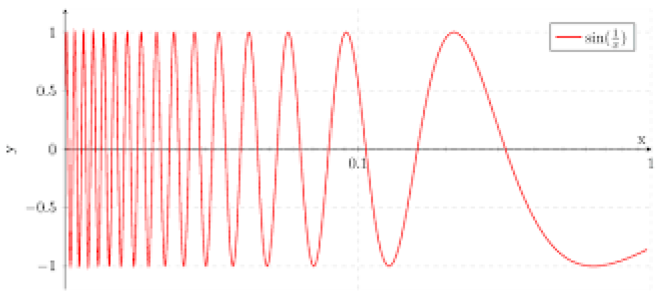


Then we see that its graph can be drawn easily in any interval without lifting pen.

However graphs are not always useful to decide the continuity of a function especially when the function oscillates a lot. We take two illustrations to understand this.

<u>First illustration</u>

Let $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



If we look at the graph of this function then we realise that as x approaches the origin the oscillations of $\sin\frac{1}{x}$ get faster and faster.

As a result the function $\sin\frac{1}{x}$ takes every single value between -1 and 1 in the interval $(-\delta, \delta)$, no matter how small is the δ. Hence; this function fails to be continuous at the origin.

<u>Second illustration</u>

Let $f(x) = \begin{cases} x, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

Note that this function oscillates so fast that it is difficult to plot its graph even in a tiniest interval. For investigating continuity of such functions we cannot rely on their graphs but have to follow the definition. Using the definition of continuity, here it is not difficult to establish that the given function is continuous only at the origin.

We now record some important results regarding continuous functions.

Theorem: If functions f and g are continuous at x, then $f+g, f-g$ And fg are continuous at x. Further if $g(x) \neq 0$, then the quotient function f/g is continuous at x.

It may be deduced from this theorem that a rational function $\frac{p(t)}{q(t)}$ is continuous everywhere except the points which are zeroes or roots of the polynomial $q(t)$.

Having established that polynomial and rational functions are continuous we finally record a result about the continuity of exponential and logarithmic functions.

Theorem:

1. If b is any positive number then the exponential function b^t is continuous on $\mathbb{R}$.
2. If b is any positive number different from 1, then the logarithmic function $\log_b t$ is continuous on $(0, \infty)$.

7. The concept of Derivative

Derivative is one of the most important tools in calculus. Its applications are not just in the field of mathematics or science but far beyond because it is associated with the study of change in one variable with respect to the other. The idea of derivative comes from the problem of determining the slope of a curve at a given point on the curve and determining the instantaneous rate of change of a variable y with respect to another variable t when the variable y is a function of t.

We now give the formal definition of derivative.

Definition: Derivative

Let $f(t)$ be a real-valued function defined on some subset D of $\mathbb{R}$. We say that $f(t)$ is differentiable at the point $c \in D$ if the domain set D contains an open interval containing the point c, and the limit

$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists. We call this limit the derivative of f at the point c, and we denote it by the symbol $f'(c)$. The process of finding the derivative is called differentiation. Some of the other common notations for the derivative $f'(c)$ of $f(t)$ at c are

$$\frac{df}{dt}(c) \text{ or } \frac{d}{dt}f(c) \text{ or } \left.\frac{df}{dt}\right|_{t=c} .$$

If $f(t)$ is differentiable at each and every point of the domain set D then we say that f is differentiable on D and $f'(t)$ represents a function on D. Note that the concept of one-sided limits can be used to define right and left derivatives also.

In economics, the concept of derivative can be used to derive the marginal revenue function from a demand function or a total revenue function because it is defined as the rate of change in the total revenue due to an arbitrarily small change in the quantity of units sold.

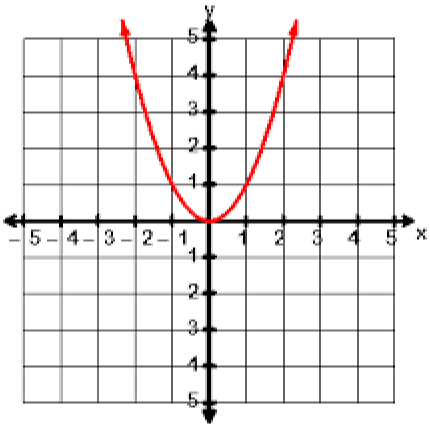
The simplest example of a differentiable function is constant function whose derivative is 0 everywhere and the identity function whose derivative is equal to 1 everywhere. We now take one example to illustrate the method of computing the derivative by its definition.

Example: Show that the function $f(x)=x^2$ is differentiable on $\mathbb{R}$ and $f'(x)=2x$.

Solution: Here

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}(2x+h)=2x.$$

We have shown here that the derivative of x^2 is $2x$. geometrically, it means that the slope of the tangent to the curve $f(x)=x^2$ at any point $x=c$, is $2c$. In particular, at origin, this slope is 0 and hence the tangent to $f(x)=x^2$ at $x=0$ is the X-axis as can be seen in the picture.



<u>Differentiability is stronger than continuity</u>

It is a very simple task to show that if a function $f(t)$ is differentiable at a point x then it is also continuous at x. But the converse of this does not hold in general. For instance, we have seen that the function $|t|$ is continuous everywhere but this function is not differentiable at $t=0$ because the limit of the quotient $\frac{|h|}{h}$ as $h\to 0$ does not exist.

8. Rules of differentiation

Let us now consider some rules of differentiation which are helpful in evaluating the derivatives.

Constant Rule

If $f(t)=c$ for all t, then $f'(t)=0$ for all t.

Scalar multiplication rule

If $f(t)$ is differentiable at $t = c$, and α is a real number then the function $(\alpha f)(t)$ is differentiable at $t = c$, and $(\alpha f)'(c) = \alpha f'(c)$.

Sum, difference, product and quotient rule for differentiation

If the functions $f(t)$ and $g(t)$ are differentiable at $t = c$, then the functions $f + g$, $f - g$, and fg are differentiable at c, and $(f \pm g)'(c) = f'(c) \pm g'(c)$ where as $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

Further if $g(c) \neq 0$, then the quotient function f/g is differentiable at c, and $(f/g)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$

Power rule for differentiation

Let $f(t) = t^p$, where p is any positive real number. If $p > 0$, then f is differentiable for all t, and $f'(t) = pt^{p-1}$. If $p < 0$, then f is defined for all non-zero t, and $f'(t) = pt^{p-1}$ for all $t \neq 0$.

The power and some other rules of differentiation discussed so far can be used collectively to prove that the polynomial and the rational functions are differentiable wherever they are defined and further, their derivatives can be easily computed using these rules.

Derivatives of exponential functions

If b is any positive number then the exponential function b^t is differentiable for all t and $\frac{d}{dt}b^t = b^t \log_e b$. In particular $\frac{d}{dt}e^t = e^t$, that is, derivative of the function e^t is the function itself.

Chain rule for differentiation

For functions f and g, suppose $g \circ f$ is a well-defined composition function on the domain of f. If f is differentiable at c and g is differentiable at $f(c)$ then the composite function $g \circ f$ is differentiable at c, and $(g \circ f)'(c) = g'(f(c))f'(c)$.

The chain rule is very useful to simplify the calculations on numerous occasions. For instance, if we have to find the derivative of $(t^2+1)^5$ then we can find this by expanding it and differentiating the expansion term by term. But this will be a very time consuming process. On the hand if we define $f(t)=t^2+1$ and $g(t)=t^5$ then $(g \circ f)(t)=(t^2+1)^5$ and by the chain rule

$$(g \circ f)'(t) = g'(f(t))f'(t) = 5(t^2+1)^4(2t) = 10t(t^2+1)^4$$

Inverse function derivative rule

Let $s=f(t)$ be an invertible function defined on an open interval containing the point $t=t_0$ and $s_0=f(t_0)$. If f is differentiable at $t_0=f^{-1}(s_0)$ and if $\left.\frac{df}{dt}\right|_{t=t_0}$ is non-zero, then $f^{-1}(s)$ is differentiable at s_0 and

$$\left.\frac{df^{-1}}{ds}\right|_{s=s_0} = \frac{1}{\left.\frac{df}{dt}\right|_{t=t_0}}$$

Since $\log_e t$ is the inverse function of e^t and $\frac{d}{dt}e^t = e^t$ using the inverse function derivative rule it may be verified that

$$\frac{d}{dt}(\log_e t) = \frac{1}{t}.$$

9. Summary

We discuss the concept of limit in detail and see how it leads to the definition of continuity and continuous functions. We learn certain examples of continuous and discontinuous functions and get familiarize with certain properties of continuous functions. The final part is dedicated to the detailed introduction of derivatives which are very useful objects in the field of science as well as economics. We learn certain examples of differentiable functions and discuss some of the rules of differentiation. These rules are very useful while investigating the differentiability or while computing the derivatives of certain functions.