[Frequently Asked Questions]

Limit, Continuity & Differentiability of a real valued function of a real variable

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Unit No. & Title:	Unit – 2 Functions
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Frequently Asked Questions

Q1. Does there exist a function on the real line which is continuous everywhere except the origin?

A1. Yes. Consider the function $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$

Q2. Does there exist a function on real line which is continuous only at $x = 2$?

A2. Yes. The function $f(x) = \begin{cases} x, & x \text{ is rational} \\ 2, & x \text{ is irrational} \end{cases}$ is continuous only at $x = 2$.

Q3. Does there exist a function on real line which is nowhere continuous?

A3. Yes. The function $f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$ is nowhere continuous because every interval contains infinitely many rational as well as irrational points.

Q4. Is every differentiable function continuous?

A4. Yes, because

$$\lim_{t\to x}(f(t) - f(x)) = \lim_{t\to x}\left(\frac{f(t) - f(x)}{t - x}\right)(t - x) = f'(x)\lim_{t\to x}(t - x) = 0$$

Q5. Is every continuous function differentiable?

A5. No, because the function $|t|$ is continuous at 0 but not differentiable.

Q6. Does there exist a function on real line which is differentiable everywhere except at $x = 2$ and $x = 3$?

A6. Yes. Consider the function $f(x) = |x - 2| + |x - 3|$.

Q7. Does there exist a function on real line which is differentiable only at $x = 0$?

A7. Yes. Consider the function $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$. Here $f'(0) = 0$ and for other values of x, $f'(x)$ does not exist because the function is not even continuous at non-zero points.

Q8. How to compute the derivative of a polynomial function

$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$**?**

A8. The derivative of a polynomial can be computed using the rule of sum, the rule of scalar multiplication and the power rule of differentiation. Accordingly, we have

$$\frac{d}{dt}\left(\sum_{k=0}^{n} a_k t^k\right) = \sum_{k=0}^{n} \frac{d}{dt}(a_k t^k) = \sum_{k=0}^{n} a_k \frac{d}{dt}(t^k) = \sum_{k=0}^{n} k a_k t^{k-1}.$$

Thus if $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ then

$p'(t) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$.

Q9. How to compute the derivative of e^{t^2}?

A9. Take $g(t) = e^t$ and $f(t) = t^2$ so that $e^{t^2} = (g \circ f)(t)$. Then using the chain rule and the fact that $g'(t) = e^t$ and $f'(t) = 2t$ we have

$$\frac{d}{dt} e^{t^2} = g'(f(t)) f'(t) = e^{t^2}(2t).$$

Q10. Does there exist a differentiable function $f(t)$ such that $f'(t) = -f(t)$ for all t?

A10. Yes; it is e^{-t}. In general the function e^{ct} for any arbitrary real constant c, satisfies the equation $f'(t) = cf(t)$ for all t.