[Academic Script]

Sequence and Series of Real Numbers

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1. Introduction

Friends, we are well familiar with the idea of summing finitely many numbers or quantities but today we are going to learn the concept of infinite sums which is needed quite often in mathematics. The idea of infinite sums is related to the concept of convergent and divergent sequences and so we shall begin with the formal definition of sequence and its limit.

2. Sequence

An infinite sequence of real numbers is a function f with domain as set of natural numbers $\mathbb{N}$ and codomain as set of real numbers $\mathbb{R}$.

Although by definition, sequence is a function f on the set $\mathbb{N}$, usually it is written as $\{x_n\}_{n=1}^{\infty}$ or as

$x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

Where x_n is the value of this function at the point $n \in \mathbb{N}$, i.e., $x_n = f(n)$. Thus intuitively, a sequence is simply a list of numbers. Our major interest here is whether a given sequence of numbers $\{x_n\}_{n=1}^{\infty}$ approaches a fixed number as n approaches infinity. Let us make this point more precise by defining the limit of a sequence.

3. Limit of a sequence

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and l be any real number. We say that the sequence $\{x_n\}_{n=1}^{\infty}$ has limit l or it converges to l, and we write,

$lim_{n\to\infty} x_n = l$ or $x_n \to l$ as $n \to \infty$,

If for every $\epsilon > 0$, there exists a positive integer n_0 such that $|x_n - l| < \epsilon$ for all $n \geq n_0$.

From the definition, we understand that a sequence $\{x_n\}_{n=1}^{\infty}$ has limit l if x_n is sufficiently close to l for all n, sufficiently large. Also it is a very simple task to show that if the limit of a sequence exists, then it is unique. If a sequence has the limit then it is called a ***convergent sequence*** otherwise it is termed as a ***divergent sequence***. Note that the simplest example of a convergent sequence is a constant sequence, i.e., $x_n = \alpha$ for all$n$. It is obvious that whether a given sequence $\{x_n\}_{n=1}^{\infty}$ is convergent or divergent depends purely on the behavior of x_n for large values of n and so the first thousand or the first billion terms of the sequence are irrelevant when we are discussing issues related to the convergence or divergence of the sequences.

Properties of Limit

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences. Then

1. $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$.
2. $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$.
3. $\lim_{n\to\infty}(x_n/y_n) = \lim_{n\to\infty} x_n \,/ \lim_{n\to\infty} y_n$, provided $\lim_{n\to\infty} y_n \neq 0$.
4. $\lim_{n\to\infty}(\alpha\, x_n) = \alpha \lim\limits_{n\to\infty} x_n \; for\; every\; real\; \alpha$.

We now discuss few theorems which are helpful in understanding the concept of convergent sequences. For this we require some definitions.

Definition: Bounded sequence

A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is said to be bounded if there exists a real number M such that

$|x_n| \le M \; for\; all\; n.$

Definition: Increasing/decreasing sequences

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then it is said to be increasing if $x_n \le x_{n+1}$ for all n. On the other hand if $x_n \ge x_{n+1}$ for all n then the sequence is said to be decreasing.

A sequence which is either increasing or decreasing is also called a **monotonic sequence.**

Theorem: Every convergent sequence is bounded.

Theorem: Every monotonic bounded sequence is convergent.

Theorem: If $x_n \to l$ as $n \to \infty$, then for every $\epsilon > 0$, there exists a positive integer n_0 such that $|x_n - x_m| < \epsilon$ for all $n, m \geq n_0$.

Let us now take a simple example.

Example: Find the limit of the sequence

$1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots \dots \dots$

Solution: Let $x_n = \frac{1}{n^2}$. clearly this sequence $\{x_n\}$ is decreasing and bounded and so it is convergent. We see that as n gets larger and larger x_n gets closer and closer to 0. Hence, we expect the limit of this sequence to be0. To prove this, given a positive number ϵ, we want to show that there exists a positive integer n_0 such that

$\frac{1}{n^2} < \epsilon$ For all $n \geq n_0$.

But it is easy to see here that if we choose n_0 to be any positive integer greater than$1/\sqrt{\epsilon}$, then our requirement is fulfilled and so we have

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

<u>About divergent sequences</u>

Consider the sequences $x_n = (-1)^n$ and $y_n = n^2$.

It is clear that none of these two sequences approach any finite number or limit no matter how far we go down the sequence, and so these two sequences are divergent sequences. Although, $x_n = (-1)^n$ and $y_n = n^2$ both are divergent sequences, there is a difference in the way they diverge. Note that the sequence $\{y_n\}$ does not converge to a finite number because its terms are approaching (positive) infinity as n tends to infinity whereas the sequence $\{x_n\}$ does not converge because its terms are bouncing around indefinitely and so cannot settle to a specific value. Such divergent sequences are also known as *oscillatory* sequences.

Sequences with limit as $+\infty$ or $-\infty$

We have given a precise meaning of the notation

$\lim_{n\to\infty} x_n = l,$

When l is a finite number but sometimes it is equally important to understand the meaning of

$\lim_{n\to\infty} x_n = +\infty$ Or $\lim_{n\to\infty} x_n = -\infty$.

Definition:

For a sequence $\{x_n\}$, if for every $M > 0$ there exists a positive integer n_0 such that

$x_n > M \, for \; all \, n \geq n_0,$

Then we say that x_n tends to (positive) infinity and write,

$\lim_{n\to\infty} x_n = +\infty.$

On the other hand if for every $M < 0$ there exists a positive integer n_0 such that

$x_n < M \, for \; all \, n \geq n_0,$

Then we say that x_n tends to (negative) infinity and write,

$\lim_{n\to\infty} x_n = -\infty.$

Note that if $\{x_n\}$ is a sequence of non-zero numbers which tends to positive or negative infinity, then

$\lim_{n\to\infty} \frac{1}{x_n} = 0.$

We now record some useful results.

Theorem: Let r be any real number and $p > 0$.

1) $\lim_{n\to\infty} n^p = \infty.$
2) If $|r| > 1$, then $\lim_{n\to\infty} |r|^n = \infty.$

3) If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$.

4) $\lim_{n\to\infty} p^{1/n} = 1$.

5) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

6) $\lim_{n\to\infty} \frac{n^r}{(1+p)^n} = 0$.

Since we are often encountered with finding the limit of quotient of polynomials $\frac{p(n)}{q(n)}$, we take an example to understand it.

Example: Determine $\lim_{n\to\infty} \frac{2n^3+n-1}{n^3+n^2+6}$.

Solution: Note that

$$\frac{2n^3+n-1}{n^3+n^2+6} = \frac{n^3\left(2+\frac{1}{n^2}-1/n^3\right)}{n^3\left(1+\frac{1}{n}+6/n^3\right)} = \frac{\left(2+\frac{1}{n^2}-1/n^3\right)}{\left(1+\frac{1}{n}+6/n^3\right)}$$

Using the properties of limit, it follows that the required limit is

$$\frac{\lim\limits_{n\to\infty}\left(2+\frac{1}{n^2}-1/n^3\right)}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}+6/n^3\right)} = \frac{2}{1} = 2.$$

The technique used in this example can be used to derive a general result about the limit of a quotient of any two polynomials.

Theorem: If $p(x)$ and $q(x)$ are polynomials with leading coefficients as $l > 0$ and $m > 0$ respectively then

a) $\lim_{n\to\infty} \frac{p(n)}{q(n)} = \frac{l}{m}$ if $p(x)$ and $q(x)$ are of same degree

b) $\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$ if degree of $q(x)$ exceeds that of $p(x)$

c) $\lim_{n\to\infty} \frac{p(n)}{q(n)} = \infty$ if degree of $p(x)$ exceeds that of $q(x)$.

In mathematics as well as economics the sequence $\left(1+\frac{1}{n}\right)^n$ has a distinguished importance and hence we need to know its behavior as n tends to infinity. Since this sequence is increasing and bounded, it is convergent and its limit is defined to be the number

e. It is an irrational number whose approximate value is 2.71828. In mathematics there are many equivalent definitions of e, but here we would like to see an economic interpretation of this number e. Note that $\left(1+\frac{1}{n}\right)^n$ is the year-end amount to the principal amount of 1 unit, if it is assumed to grow at 100% per annum and if the interest is compounded n times in a year at regular intervals. With this interpretation, e is simply the year-end amount to the principal amount of 1 unit, if it is assumed to grow at 100% per annum and if the interest is compounded continuously?

In economics, we often deal with exponential, logarithmic and polynomial functions and on certain occasions it is important to compare the growth of these functions with each other. The following result highlights this matter.

Theorem: Let t be any positive number. Then

1) $\lim_{n\to\infty}\frac{\log n}{(1+t)^n}=0.$
2) $\lim_{n\to\infty}\frac{p(n)}{(1+t)^n}=0$ where $p(n)$ is a polynomial in n.
3) $\lim_{n\to\infty}\frac{\log n}{p(n)}=0$ where $p(n)$ is a polynomial in n.

It can be inferred from these three results that the growth of the exponential function is much faster as compared to the growth of logarithmic or polynomial functions. Further, the growth of the polynomial function is faster than that of logarithmic function.

4. Series of numbers

Given a sequence $\{x_n\}$ of real numbers, we associate with it the infinite sum or series denoted by $\sum_{n=1}^{\infty}x_n$. Our goal here is to assign a precise meaning to the symbol $\sum_{n=1}^{\infty} x_n$. Intuitively, we understand that $\sum_{n=1}^{\infty} x_n$ means the value of the finite sum $\sum_{n=1}^{k} x_n$ for larger and larger values of k and this intuition leads to the following idea of convergence of series.

About Convergent and Divergent series

Given a sequence $\{x_n\}$, consider the finite sums of the type $S_k = \sum_{n=1}^{k} x_n$, for $k \in \mathbb{N}$. Then we say that the series $\sum_{n=1}^{\infty} x_n$ converges to a number l and write $\sum_{n=1}^{\infty} x_n = l$ if the sequence $\{S_k\}$ converges to l as $k \to \infty$. If this does not happen then we say that the series is divergent and the symbol $\sum_{n=1}^{\infty} x_n$ has no meaning. However, if $\lim_{k\to\infty} S_k$ is positive or negative infinity, then we say that the series $\sum_{n=1}^{\infty} x_n$ diverges to infinity and write $\sum_{n=1}^{\infty} x_n = \infty$ or $\sum_{n=1}^{\infty} x_n = -\infty$ respectively.

The sequence $\{S_k\}$ is often called the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$. We have earlier made a point that the convergence or divergence of any sequence is not affected by the first million or the first billion terms of the sequence; so this applies to the sequence $\{S_k\}$ and consequently the convergence or divergence of any series is not affected by the first million or first billion terms of the series.

Notice that if the series $\sum_{n=1}^{\infty} x_n$ converges to a number l then

$\lim_{k\to\infty} x_k = \lim_{k\to\infty} (S_k - S_{k-1}) = l - l = 0.$

Thus for the series$\sum_{n=1}^{\infty} x_n$, if $\lim_{k\to\infty} x_k \neq 0,$then we can immediately conclude that the series is not convergent. On the other hand if $\lim_{k\to\infty} x_k = 0$, then the series may not be necessarily convergent. For this, we may consider the example of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. it can be worked out that if $S_k = \sum_{n=1}^{k} \frac{1}{n}$, then $S_{2^k} \geq \frac{k+2}{2}$ and so the sequence $\{S_k\}$ turns out to be unbounded. This results into $\{S_k\}$ being divergent and hence the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is not convergent. In fact, we have $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$. this can also be interpreted as; given any positive number M we can find a positive integer k such that $\sum_{n=1}^{k}\frac{1}{n}>M$.

5. Basic properties of the series

1) If the series $\sum_{n=1}^{\infty}x_n$ and the series $\sum_{n=1}^{\infty}y_n$ are convergent then the series $\sum_{n=1}^{\infty}(x_n \pm y_n)$ is convergent and

$$\sum_{n=1}^{\infty}(x_n \pm y_n) = \sum_{n=1}^{\infty}x_n \pm \sum_{n=1}^{\infty}y_n.$$

2) If the series $\sum_{n=1}^{\infty}x_n$ converges and $\alpha \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty}(\alpha x_n)$ converges and

$$\sum_{n=1}^{\infty}\alpha x_n = \alpha\sum_{n=1}^{\infty}x_n.$$

6. Geometric series

Let r be a real number. Then the series $\sum_{n=1}^{\infty}r^n = r+r^2+r^3+\cdots$ is called the geometric series. The test for the convergence of geometric series is very simple. If $|r|<1$ then it is convergent and further $\sum_{n=1}^{\infty}r^n = \frac{r}{1-r}$ whereas if $|r|\geq 1$ then it is divergent.

Such type of series occurs when we think about the decimal expansions of some rational numbers. For instance, if we consider the number $1/3$ its decimal expansion is $0.3333\ldots..$ which may be written as

$$3\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right)=3\left(\frac{1/10}{1-1/10}\right)=\frac{3}{9}=\frac{1}{3}.$$

7. Telescoping series

A telescoping series is a series whose partial sums eventually have fixed number of terms after cancellation. One popular example of such a series is

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} \quad \text{or more generally} \sum_{n=1}^{\infty} \frac{1}{(n+l)\,(n+p)} \text{ where } l < p.$$

Here we may write

$$\sum_{n=1}^{\infty} \frac{1}{(n+l)\,(n+p)} = \frac{1}{p-l}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n+l} - \frac{1}{n+p}\right)\right)$$

It is very easy to determine the convergence of the series which are geometric or telescoping but this is not always the case for Arbitrary series. So we now introduce some of the tests which help us in determining the convergence or divergence of a given series.

8. Comparison test

1) If $|x_n| \le y_n$ for $n \ge n_0$, and if $\sum y_n$ converges, then $\sum x_n$ converges.
2) If $x_n \ge y_n \ge 0$ for $n \ge n_0$, and if $\sum y_n$ diverges then $\sum x_n$ diverges.

The comparison test is very useful especially if we are familiar with certain convergent or divergent series of non-negative numbers.

9. Cauchy's Condensation test

Let $\{x_n\}$ be a decreasing sequence of non-negative terms. Then the series $\sum x_n$ converges if and only if the series $\sum_{n=1}^{\infty} 2^n x_{2^n}$ converges.

In view of the geometric series test, the Cauchy's condensation test now immediately gives some nice results.

Results

1) The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$.
2) The series $\sum \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Using these results we now work out an example based on the comparison test.

Example: Determine the convergence or divergence of the series $\sum \sin n^2/n^2$ and $\sum \frac{\log n}{n}$.

Solution: Let $x_n = \sin n^2/n^2$ and $y_n = 1/n^2$. since $|x_n| \leq y_n$ for all $n \geq 1$ and because we know that $\sum 1/n^2$ is a convergent series, it follows from the comparison test that the series $\sum \frac{\sin n^2}{n^2}$ converges. For the second series we take $x_n = \log n/n$ and $y_n = 1/n$. Note that $x_n \geq y_n \geq 0$ for large values of n and further the harmonic series $\sum 1/n$ is divergent. So the comparison test implies that the series $\sum \frac{\log n}{n}$ diverges.

10. The Alternating series test

Let $\{x_n\}$ be a decreasing sequence of non-negative terms. Then the series $\sum(-1)^n x_n$ converges if $\lim_{n\to\infty} x_n = 0$.

Recall that $\sum \frac{1}{n^p}$ is divergent for $0 < p \leq 1$ follows from the Cauchy condensation test. But if we alternate the sign of the terms of this series then due to the alternating series test we deduce that $\sum(-1)^n/n^p$ is convergent for any $p > 0$. this result indicates that sometimes the convergence of a series is greatly influenced by the cancellations involved in the series.

The comparison test suggests that a series converges if the rate of convergence of its terms to zero is high enough. But it is not always possible to judge this rate of convergence and in such scenarios, sometimes the ratio and the root tests are useful. So we discuss these tests one by one.

The Ratio test

For the series$\sum x_n$, suppose that $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = l.$ then

1) $\sum x_n$ converges if $l < 1$;

2) $\sum x_n$ diverges if $l > 1$;

3) the test is inconclusive if $l = 1$.

The ratio test can be applied to conclude that the series $\sum x^n/n!$ converges for every real x. Further; it can also be used to establish that the series $\sum n!/n^n$ converges.

The ratio test is mainly used when the general term of the series involve factorials. Some of the series where this test is inconclusive are $\sum 1/n^p$ and $\sum \frac{n^2}{n^2+2n}$.

The Root test

For the series $\sum x_n$, suppose that $\lim_{n\to\infty} |x_n|^{1/n} = l$. Then

1) $\sum x_n$ converges if $l < 1$;

2) $\sum x_n$ diverges if $l > 1$;

3) the test is inconclusive if $l = 1$.

Here we are assessing the value of the n^{th} root and so the root test is mainly used when the general term of the series has powers. The following rules for non-exponentials are also sometimes useful while applying the root test.

1) $\lim_{n\to\infty} c^{1/n} = 1$ for every positive constant c.

2) $\lim_{n\to\infty} (\log n)^{1/n} = 1$.

3) $\lim_{n\to\infty} (n^p)^{1/n} = 1$ for every positive exponent p.

4) $\lim_{n\to\infty} (n!)^{1/n} = \infty$.

Example: Test the convergence or divergence of the series $\sum(-1)^n n^2/(1+n)^n$ and $\sum \frac{2^{n^2}}{n^n}$.

Solution: Let $x_n = \frac{(-1)^n n^2}{(1+n)^n}$. Then

$$\lim|x_n|^{\frac{1}{n}} = \lim\frac{\left(n^{\frac{1}{n}}\right)^2}{1+n} = \lim\frac{1}{1+n} = 0 < 1.$$

Hence by the root test the series $\sum(-1)^n n^2/(1+n)^n$ is convergent. For the second series we observe that

$$\lim|x_n|^{\frac{1}{n}} = \lim\frac{2^n}{n} = \infty$$

And hence the series $\sum\frac{2^{n^2}}{n^n}$ is divergent.

11. Summary

We give a formal definition of a sequence and its limit and understand the concepts related to the convergence and divergence of sequences. Later, we use the concept of a limit of a sequence to define the infinite sums or series. The problem of determining whether a given series is convergent or divergent is not always easy and for this various tests are introduced and studied in this module.