[Academic Script]

Elementry Theory of Functions

Subject:	Business Economics
Course:	B. A. (Hons.), 1st Semester, Undergraduate
Paper No. & Title:	Paper – 202 Mathematics for Business Economics
Unit No. & Title:	Unit – 1 Basic Concepts
Lecture No. & Title:	Lecture – 2 Elementry Theory of Functions

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1. Types of relation in Economics:

For that we shall have to first try to understand ordered pairs and Cartesian product of sets. Let us begin with:

Ordered pairs:

We have noted that the order of elements is not important in a set. But when we want order to be important, we write (a, b) instead of $\{a, b\}$ which is the notation for set. We call it an ordered pair.

We emphasize that $(a, b) \neq (b, a)$ unless $a = b$.

Also $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

And note well that $(a, b) \neq (c, d)$ if and

only if either $a \neq c$ or $b \neq d$.

Example: To record weight of each student of a class of thirty students one can form ordered pairs

(Roll. No. of student, his weight in specified units.)

Thus (1, 50) would mean that student having Roll. No. 1 has weight 50 units. And (25, 60) would mean that student having Roll. No. 25 has weight 60 units. Clearly here (20, 25) and (25, 20) are different and have here totally different meanings.

Cartesian Product: If two sets X and Y are given their Cartesian Product which is denoted by $X \times Y$ is defined as:

$X \times Y = \{(\mathrm{a}, \mathrm{b}) | \mathrm{a} \in X \text{ and } \mathrm{b} \in Y\}$

Example:

If $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$ then,

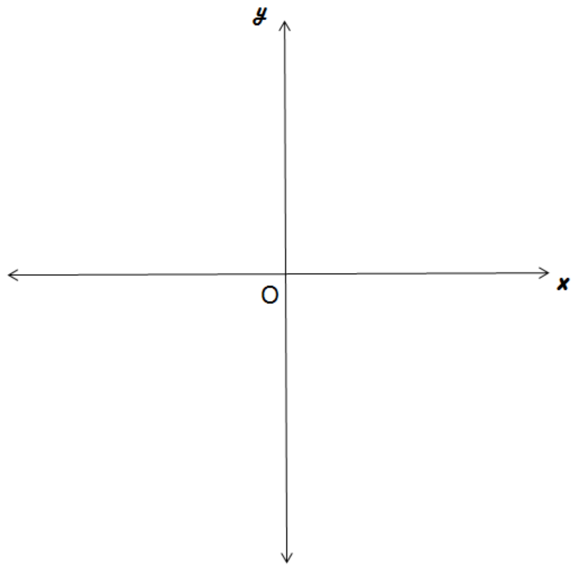
$X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

$Y \times X = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$

It is clear that $X \times Y \neq Y \times X$.

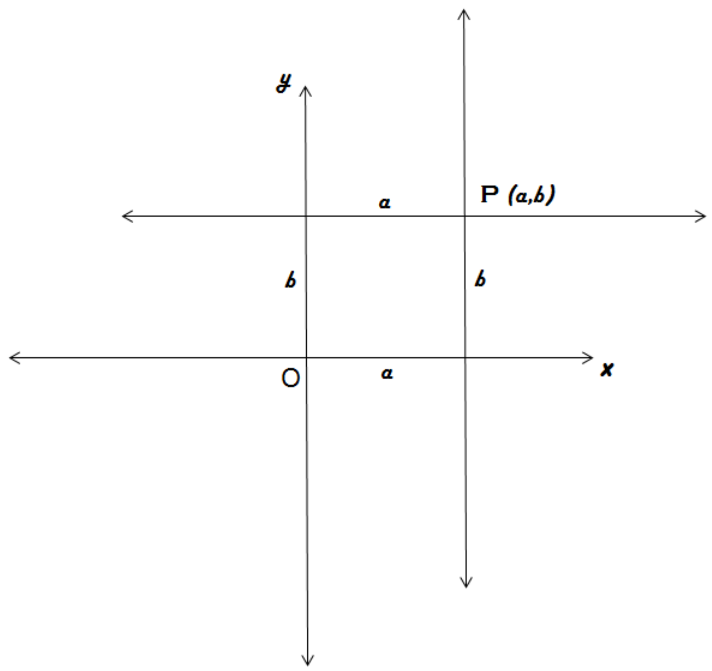
As we know that the points on a line can be associated with real numbers. Similarly points on the plane can be associated with elements of $\mathbb{R} \times \mathbb{R}$.

Rectangular Coordinate Plane: A plane with two orthogonal lines meeting at a point O, which will be referred to as the origin, is called the Rectangular Coordinate Plane. If necessary after rotating the plane, we visualize this plane as shown in the figure. The horizontal line through origin is most often named as the x-axis and the vertical line through O is named as the y-axis. Further the right-half x-axis is called the positive x-axis and the left-half x-axis is called the negative x-axis. Similarly upper-half of y-axis is called the positive y-axis and the lower-half y-axis is called the negative y-axis.

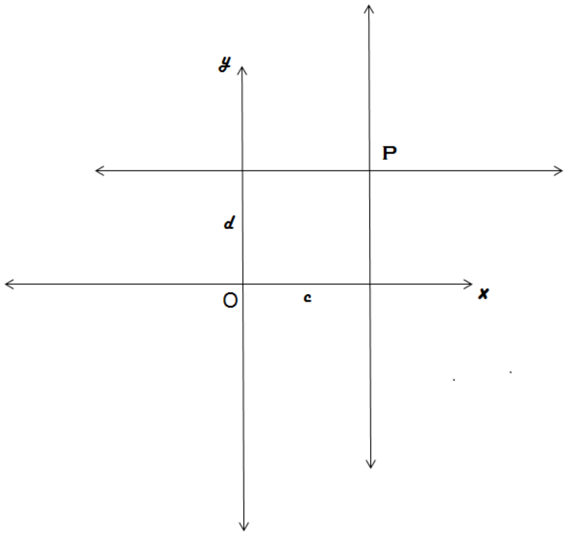


There is a one-to-one correspondence between the set of ordered pairs of real numbers i.e. $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, and the set of points of a plane, when we consider the plane as a Cartesian coordinate plane.

Suppose ordered pair of real numbers (a, b) i.e. that is when an element of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is given, we associate with this ordered pair a point P in the Cartesian Coordinate Plane by fixing real number a on the x-axis and then fixing the real number b on the y-axis. Thus P is at the intersection of two lines which are parallel to coordinate axes i.e. x-axis and y-axis as shown in the figure. In other words we associate the point P which is obtained by moving distance a along the x-axis and then moving distance b along the y-axis.

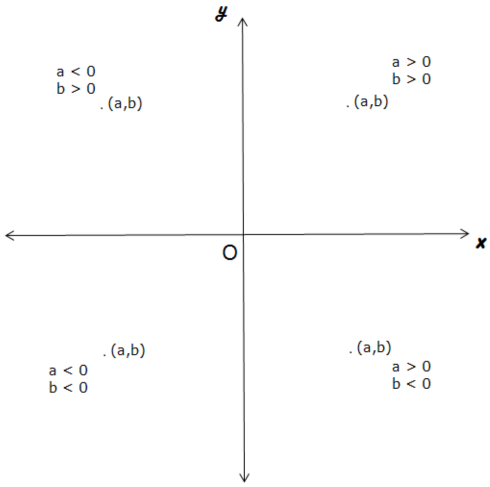


Also the other way, when a point P is given in the Cartesian Coordinate Plane, we can associate with P, an ordered pair of real numbers (c, d) i.e. an element of $\mathbb{R}^2$ by drawing two lines through P which are parallel to coordinate axis, and then associating with the intersection points, the real numbers c and d on x-axis and y-axis respectively as shown in the figure:



Thus we can establish one-to-one correspondence with the set of all ordered pairs of real numbers and the set of points on a Plane which is called Cartesian Coordinate Plane.

The Cartesian Coordinate Plane is divided into four quadrants:



The first quadrant consists of ordered pairs (a, b) where both the coordinates$a$ and b are positive. The second quadrant consists of ordered pairs (a, b) where the first coordinate a is negative and the second coordinate b is positive. The third quadrant consists of ordered pairs (a, b) where both the coordinates a and b are negative. And the fourth quadrant consists of ordered pairs (a, b) where the first coordinate a is positive and the second coordinate b is negative.

2. Relation

A relation R is a subset of $X \times Y$. Any such subset establishes a relation between a certain points y of set Y and certain points x of the set X.

Examples:

1. If $X = \{1,2,3\}$ and $Y = \{a,b,c\}$ then,

$$R = \{(1,a),(2,b)\}$$

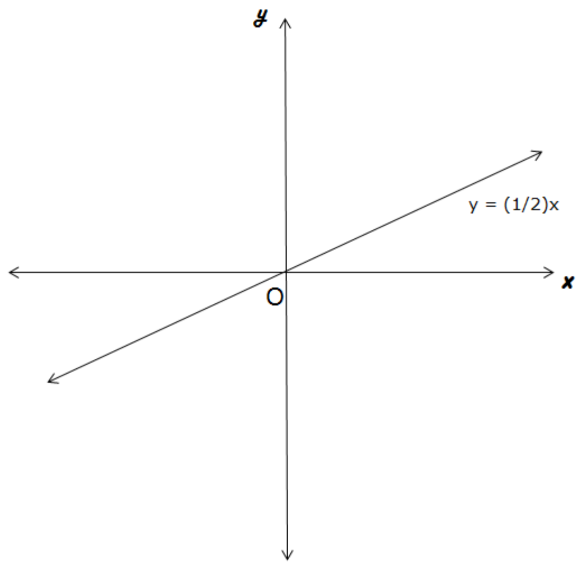
   is a relation.

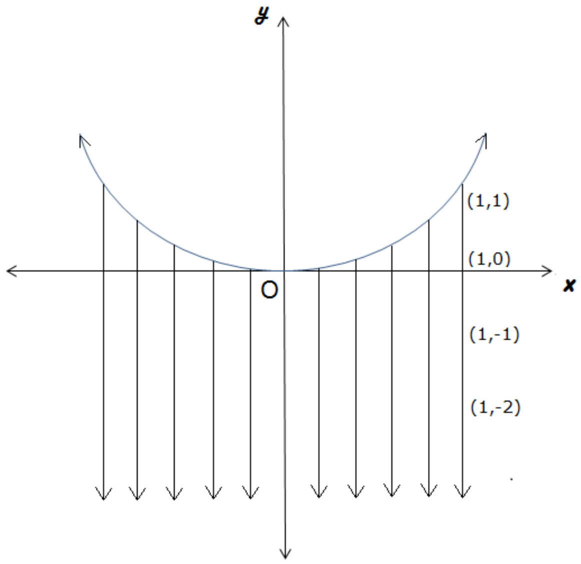
2. Suppose X = is the set of all persons who were alive in world on 1st Jan 2000 and Y = the set of all women who were alive in this world on 1st Jan 2000. Then,

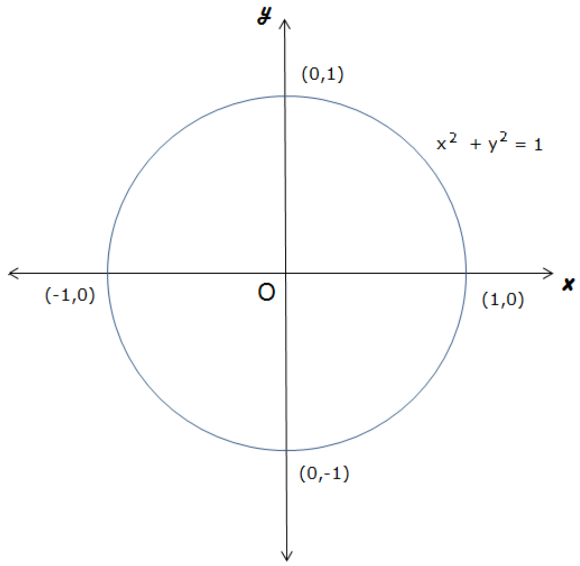
$$S = \{(x,y) \in \boldsymbol{X} \times \boldsymbol{Y} \mid y \text{ is a mother of } x\}$$

   is a relation.

3. The set $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = (1/2)\, x\}$ is a relation.
4. The set $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y \leq x^2\}$ is also a relation.
5. The set $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$ is a relation.







Remarks:

1. Suppose a relation R that is a subset R of $X \times Y$ is given. Note that it is not necessary that for every element x in X, there is some y in Y such that (x, y) is in our relation R. In our Examples, Example Number 1, 2 and 5 are such examples. In example 1, no element of the set Y is related to 3. In example 2, if x a person who was alive on 1st Jan 2000 but whose mother had died and was not there on that day then for such x in X there is no y in Y such that $(x, y) \in R$. In example 5, it is very clear that if we take x greater than 1 or less than -1 then for such x there is no

real number y such that $x^2 + y^2 = 1$ and therefore there is no y such that $(x, y) \in R$.

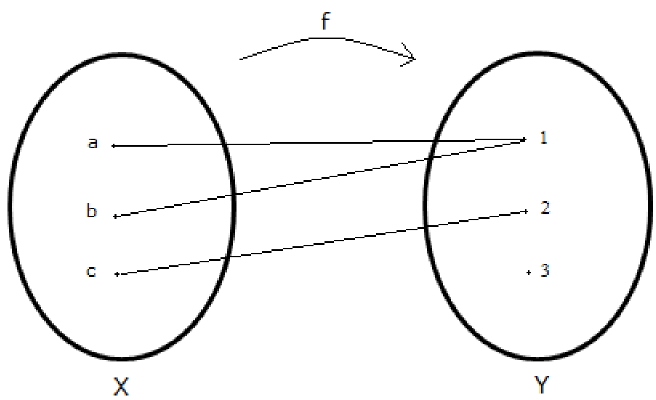
2. Relation R may be such that for some x, there are more than one related y values. In this case the relation R is called one-many. In our Examples, Example Number 4 and 5 are such examples. In example 4, there are infinitely many values of y which are related to $x = 1$. Here $(1,1)$, $(1,0)$, $(1,-1), (1,-2) \ldots \in R$.

3. Function

When a relation R i.e. a subset of $X \times Y$ is such that for each x in X there is uniquely related y, we say that this relation defines a function. In this case we write:

$y = f(x)$ whenever (x, y) is in R

Often a function is visualized as:



Here $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$ and $R = \{(a, 1), (b, 1), (c, 2)\}$.

Remarks:

1. Functions are also called mapping or transformations.
2. Given a relation R, different x may have same related values. That is (x_1, y) and (x_2, y) may be in R. E.g.

$S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = |x|\}$ here (1,1) and (-1,1) are in S similarly (2,2) and (-2,2) are also in R.

3. We should also note that every function is a relation, but a relation need not be a function.The earlier considered set

$$S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$$

is a relation. But this relation is not a function. This relation fails to be a function for two reasons:

i. With $x = 2$ there is no related value y.

ii. With $x = 0$ there are more values related. Here (0, 1) and (0, -1) are in S.

4. Function is also specified by the notation $f: X \to Y$ where for each x in X a unique value $f(x)$ is specified and clearly then $\{(x, f(x)) | x \in X\} \subseteq X \times Y$ is a function in the strict sense of the definition. Conversely if relation $R \subseteq X \times Y$ is given that is also a function, then $f: X \to Y$ is defined by writing $f(x) = y$ if (x,y) is in R.

When a function $f: \mathbb{R} \to \mathbb{R}$ is given, graph of f, is nothing but, a function looked upon as a relation

$$\{(x, f(x)) | x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$$

which is a subset of the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

4. Types of Functions

Constant function: Functions of the type $y = f(x) = 10$ are called constant functions. They do not change their values. Whatever may be $x, y = f(x)$ is constantly equal to 10 here. The graph of such a function is a horizontal line.

Polynomial function: The function of the type

$y = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

is called the polynomial function, note that here $a_0, a_1, \ldots a_n$ are fixed real numbers i.e. they are real constants. Depending on the different values of n, the functions that we get are given definite names. These names are summarized in the table:

The value of n	The corresponding polynomial	The given name
0	$y = f(x) = a_0$	Constant function
1	$y = f(x) = a_0 + a_1 x$	Linear function
2	$y = f(x) = a_0 + a_1 x + a_2 x^2$	Quadratic function
3	$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$	Cubic function

Rational functions:

A function of the type $y = \frac{1+x+x^2}{1+x+x^2+x^3}$ wherein y is given as a ratio of two polynomial functions is called the rational function and if nothing is mentioned it is supposed to be defined on the set of all real numbers minus the set of real numbers where denominator polynomial is zero.

If $f: X \rightarrow Y$ is a function, the set X is called the domain of the function, the set Y is called the co-domain and the set of values of the function i.e.

$$\{f(x) | x \in X\} \subseteq Y$$

is called the range of the function.

Example: If the total cost C of a firm per day as a function of the total daily output Q is given by:

$$C = C(Q) = 100 + 8Q \text{ where } Q \in \{ Q \mid 0 \leq Q \leq 150 \}$$

Then the domain of the cost function C is $\{ Q \mid 0 \leq Q \leq 150 \}$. And the range of the cost function is $\{Q \mid 100 \leq Q \leq 1300\}$.

Remarks:

1. Note that the from the cost function that we have considered cost value C can be found for any Q even for negative Q. However in the given situation one has to specify the domain or should be well-understood without any ambiguity. Then only one finds out the range.

2. In modern mathematics one always specifies the co-domain. We say that the function is onto if the range is the co-domain. The phrase "surjective function" is also used for "onto function". That is $f: X \to Y$ is onto if for each y in Y there exists x in X such that $f(x) = y$.

3. When the values of the function are different for different values in the domain we say that the given function is one-to-one, one-one, 1-1 or injective.

4. In general notations it means that: $f: X \to Y$ is one-to-one or injective if

$$for\ all\ x_1 \neq x_2\ in\ X\ we\ must\ have\ f(x_1) \neq f(x_2)$$

In other words it means that:

$$f(x_1) = f(x_2)\ implies\ \ x_1 = x_2\ \ for\ all\ \ x_1, x_2\ \ in\ X$$

The last alternative phrase is very handy when one wants to prove that a given function is injective.

5. A function which is both injective as well as surjective is called a bijective function. In other words a bijective functions is both 1-1 and onto.

Example: The example that we considered earlier i.e.
$C = C\,(Q) = 100 + 8Q$; is injective as:
$C(Q_1) = C(Q_2)$ implies $100 + 8Q_1 = 100 + 8Q_2$ implies $Q_1 = Q_2$

Monotonic Functions: If a function produces successively larger and larger values when we take successively larger and larger values from the domain, we say that the function is monotonically increasing or simply increasing i.e. a function is increasing if

$x_1 < x_2 \; implies\; that\; f(x_1) < f(x_2)$

Similarly,
If a function produces successively smaller and smaller values when we take successively larger and larger values from the domain, we say that the function is monotonically decreasing or simply decreasing i.e. a function is decreasing if
$x_1 < x_2 \; implies\; that\; f(x_1) > f\,(x_2)$

Inverse Function:

Suppose $y = f\,(x)$ is a function and if we write:

$$x = f^{-1}(y) \text{ if and only if } y = f(x)$$

then clearly f^{-1} will be defined on the range of f and more over it will be one-one iff f is one-one and it also will be one-many and therefore not a function iff f is many-one. So if we want f^{-1} to be a function f has to be one-one function. It will be defined on the range of f.

Note that increasing or decreasing functions are one-one functions and therefore inverse from the range of the function exists.

Here are some examples on inverse functions:

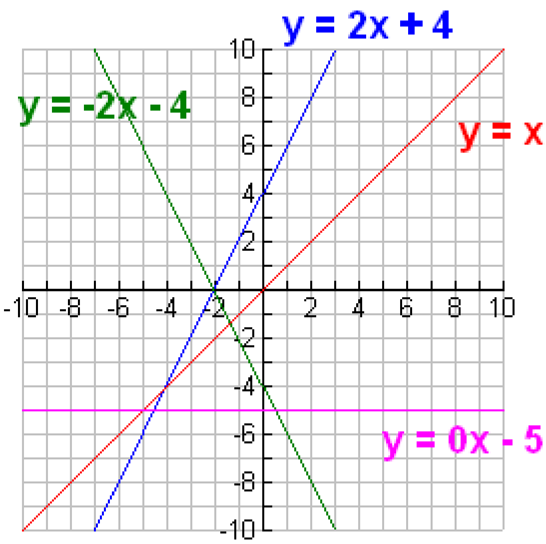
Examples:

1. If $y = f(x) = 1 + 3x$ then $x = f^{-1}(y) = (y-1)/3$
2. Note that $y = f(x) = x^2$ does not have an inverse function as it is many-one on $\mathbb{R}$.
3. However if we consider $y = f(x) = x^2$ for only non-negative x then it is one-to-one and its inverse from the non-negative real numbers to non-negative real numbers exist and is given by $x = f^{-1}(y) = \sqrt{y}$ the non-negative square-root of y.

Some of the very useful functions along with their graphs are displayed here:

Linear function:

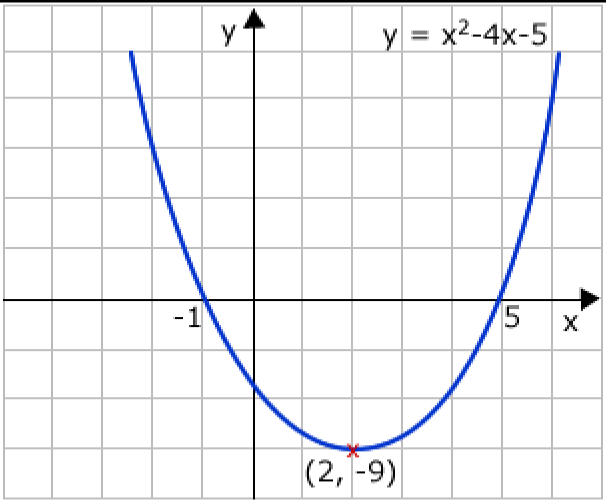
$f: \mathbb{R} \to \mathbb{R}$, defined by $y = f(x) = a_0 + a_1 x$, is called a linear function. Its graph, for some chosen values of the constants a_0,

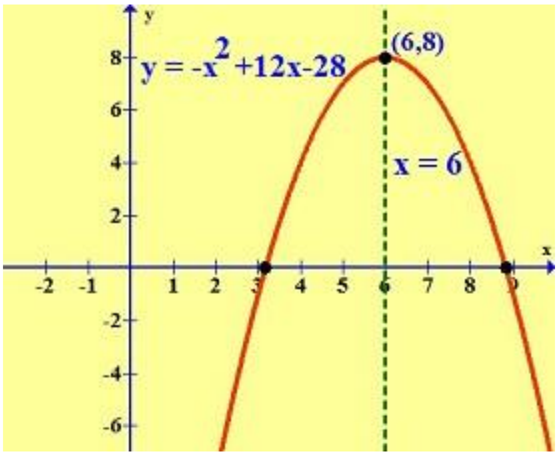


and a_1 are as:

Quadratic function:

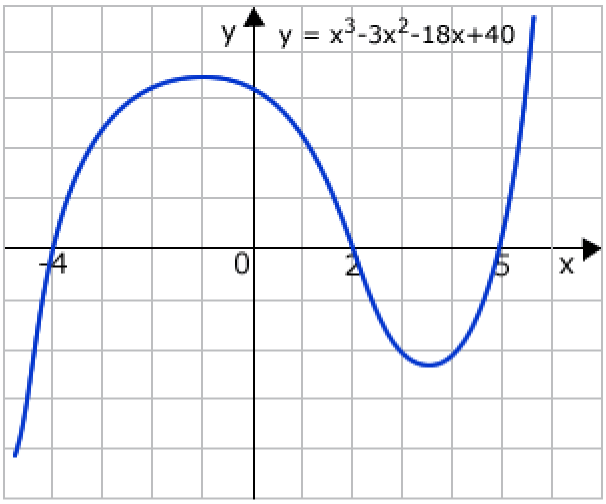
$f: \mathbb{R} \to \mathbb{R}$, defined by $y = f(x) = a_0 + a_1 x + a_2 x^2, a_2 \neq 0$ is called a quadratic function. Its graph, for some chosen values of the constants a_0, a_1 and a_2 are as:

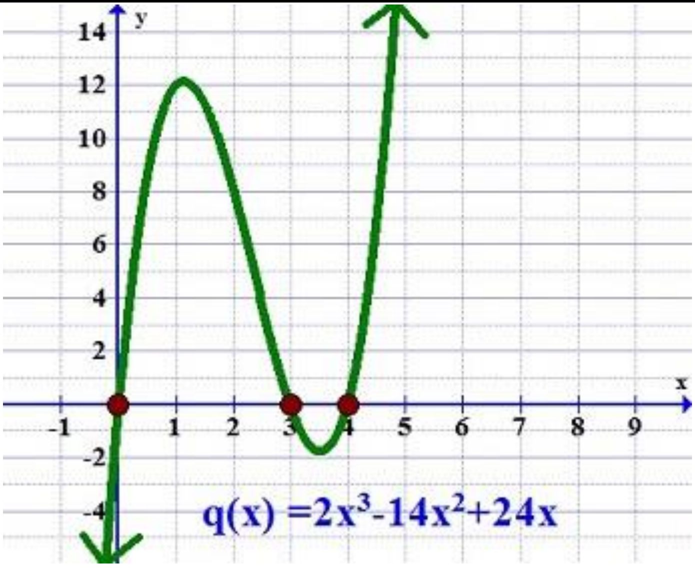




Cubic function:

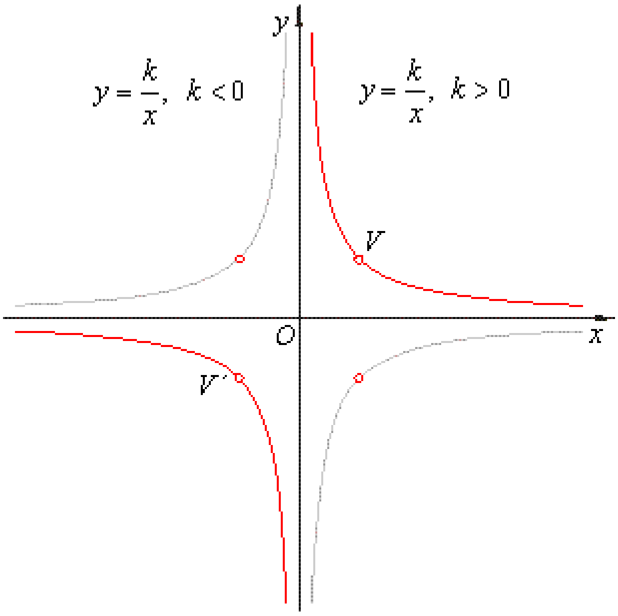
$f:\mathbb{R}\to\mathbb{R}$, defined by $y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3, a_3 \neq 0$ is called a Cubic function. Its graph, for some chosen values of the constants a_0, a_1, a_2 and a_3 are as:





Rectangular Hyperbola:

$f:\mathbb{R}-\{0\}\to\mathbb{R}$, defined by $y=f(x)=\frac{k}{x}$ is called the rectangular hyperbola. Depending on k being positive or negative we get the graphs as:

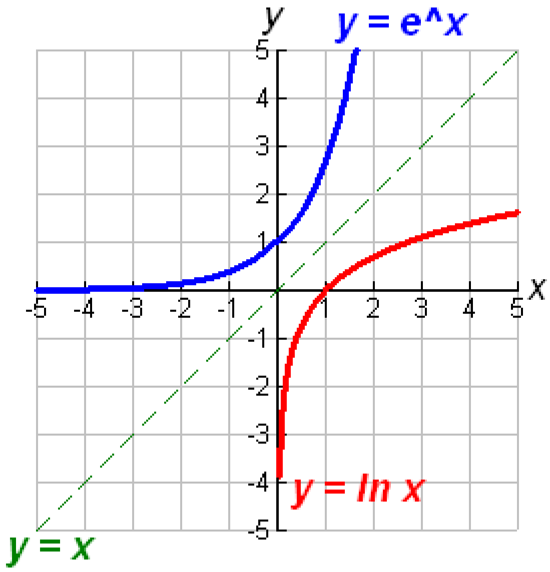


Exponential and Logarithmic functions:

$f:\mathbb{R}\to\mathbb{R}$, defined by $y=f(x)=a^x$ $a>1$ and in particular when $a=e$, we get what is known as exponential function. Some graphs for different values of x are shown in the figure.

$f:\mathbb{R}^+\to\mathbb{R}$, defined by $y=f(x)=\log_a x$ $a>1$ and in particular when $a=e$ it is called the natural logarithm and is denoted by$y=f(x)=\ln x$. Note that these functions that is $f(x)=a^x$ and

$f(x) = \log_a x$ are inverse functions of each other. Let us have a look at their graphs:



5. Summary

Starting with the definition of ordered pairs and Cartesian product of sets we have gone into the discussion of relations and functions. We also have introduced Cartesian Coordinate Plane. Discussion about graphs of relations and functions was carried out at a great length showing how they can be plotted when they are given as subsets of the plane $\mathbb{R} \times \mathbb{R}$. Certain definite types of functions including increasing and decreasing functions and their graphs were also discussed. Inverse functions were also the part of the discussion.